

# THEORETICAL ASPECTS OF THE FLOW OF WATER THROUGH ANISOTROPIC UNSATURATED SOILS

Aristides C. LIAKOPOULOS Ph. D.

*Assistant Professor of Civil Engineering at the American University of Beirut*

## ABSTRACT

The description of flow of water through an unsaturated soil is governed by the same general laws that govern all kinds of fluid motion. The laws expressed in terms of mathematical equations are the continuity equation, the equation of state and the dynamical equation of motion. The flowing water has to satisfy the above three equations at any point of the considered soil and at any time. This leads to a non-linear partial differential equation in which the independent variables are the time and the position in space and the dependent variable is the pressure of the water or the volumetric water content. The equation is derived in cartesian and cylindrical coordinates by considering a cubical and a sector element of water conducting soil respectively.

The governing equation is of such a nature that a solution exists for time  $t > 0$  and is uniquely determined if two relationships are defined together with the specified state of the system at the initial time  $t = 0$  and at the boundaries. The two required relations are these of pressure versus hydraulic conductivity and pressure versus volumetric water content.

Because of the strong non-linearity in its terms, the equation can be solved for various initial and boundary conditions by approximating the differentials with finite differences at discrete points in the solution domain. Either a relaxation or an iteration procedure can be used.

## INTRODUCTION

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity  $\vec{v}$  and of any two thermodynamic quantities pertaining to the fluid; these two quantities can be taken to be the pressure  $p$  and the density  $\rho$ . The above quantities  $\vec{v}$ ,  $p$  and  $\rho$  are functions of the cartesian coordinates  $x$ ,  $y$ ,  $z$  and of the time  $t$ , or functions of the cylindrical coordinates  $r$ ,  $\theta$ ,  $z$  and  $t$ .

As it is well known, all the thermodynamic quantities are determined by the values of any two of them (i.e.  $p$  and  $\rho$ ) together with the equation of state. That means that if we are given five quantities the state of the moving fluid is completely determined. The five quantities are the three components of the velocity  $\vec{v}$ , the pressure  $p$  and the density  $\rho$ . Accordingly a complete system of equations of fluid dynamic should be five equations. These are the three Navier-Stokes equations which are the hydro-dynamical restatement of Newton's first law of motion, the equation of continuity and the thermodynamic equation of state.

The dynamical equation of motion expressing the balance between body forces, viscosity forces and inertia forces is given by the Navier-Stokes equation:

$$-\frac{\partial (p + \rho gh)}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)$$

where  $\mu$  is the viscosity of the fluid. The other two force components in the  $y$  and  $z$  directions may be written in the abbreviated form:

$$-\frac{\partial (p + \rho gh)}{\partial y} + \mu \nabla^2 v_y = \rho \frac{Dv_y}{Dt}$$

and

$$-\frac{\partial (p + \rho gh)}{\partial z} + \mu \nabla^2 v_z = \rho \frac{Dv_z}{Dt}$$

Assuming that the flow through the pores of soil is laminar, the inertia forces become negligible and the Navier-Stokes equation can be reduced to Darcy's equation stated in the form:

$$\vec{v} = \frac{k\rho}{\mu} \vec{F}$$

where  $\vec{F}$  is the force equal to the negative of the potential gradient

$$\vec{F} = -\nabla\varphi$$

Since the potential is the sum of gravitational and pressure potential

$$\varphi = gz + \frac{p}{\rho}$$

the dynamic equation of motion may be written as

$$\vec{v} = -\frac{k\rho}{\mu} \nabla\varphi = -\frac{k\rho}{\mu} \nabla\left(gz + \frac{p}{\rho}\right) \quad (1)$$

The symbol  $k$  stands for the intrinsic permeability of the soil and is defined as  $k = cd^2$  where  $c$  is a dimensionless factor of proportionality depending on the porosity, packing, shape of particles etc, and  $d$  is a length such as the mean grain diameter characterizing the size scale of the openings of the soil. It is interesting to note that equation (1) can be written as:

$$\vec{v} = -\frac{k\rho}{\mu} \left( \nabla gz + \nabla \frac{p}{\rho} \right) = -\frac{k\rho}{\mu} \left( -\vec{g} + \frac{1}{\rho} \nabla p \right)$$

or

$$\vec{v} = +\frac{k}{\mu} (\rho\vec{g} - \nabla p) \quad (2)$$

and in cases where  $k$  and  $\mu$  are variables:

$$\vec{v} = +\frac{k\rho g}{\mu} - \nabla \frac{kp}{\mu} \quad (3)$$

One should admit that equation (2) is easier to treat analytically; it seems that it gives a very satisfactory description of experimental results and also can be used easily for immiscible multiple phase flow. That may explain the fact that equation (3) is not used in research of problems in porous media flow. The difference between equations (1) and (2) depends on whether a variation of the density is being considered or not.

The term  $gz$  is the gravitational potential and the term  $p/\rho$  is the pressure potential or capillary potential for unsaturated soils. The symbol  $g$  denotes the acceleration of gravity, whereas  $\vec{g}$  denotes the gravity vector inducing thus the direction of the gravity.

Equation (1) can be written in terms of head ( $h$ ). Since:

$$\varphi = gh$$

one has:

$$h = \frac{\varphi}{g} = z + \frac{p}{\rho g} \quad (4)$$

and therefore the gradient of the potential is written as:

$$\nabla \phi = \nabla gh = g \nabla h$$

Equation (1) therefore becomes:

$$\vec{v} = - \frac{k \rho g}{\mu} \nabla h = - \frac{k \rho g}{\mu} \nabla \left( z + \frac{p}{\rho g} \right) \quad (5)$$

where  $z$  is referred to as the elevation head and  $p/\rho g$  as the pressure head or capillary head. Defining:

$$K = \frac{k \rho g}{\mu} = \frac{k \gamma}{\mu}$$

where  $\gamma$  is the unit weight of the water and  $K$  the hydraulic conductivity of the soil (with dimensions of velocity), the dynamic equation of motion becomes:

$$\vec{v} = - K \nabla h \quad (6)$$

The above dynamic equation of motion has to follow certain restrictions according to the continuity equation:

$$\frac{\partial(\rho c)}{\partial t} + \rho \nabla \cdot \vec{v} + \vec{v} \nabla \rho = 0 \quad (7)$$

where  $c$  is the volumetric water content.

Designating the components of vector velocity at the point  $(x, y, z)$  as  $v_x, v_y, v_z$  and considering the density of water as constant, equation (7) becomes:

$$\frac{\partial c}{\partial t} = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad (8)$$

where  $\vec{v}$  signifies the volume flux per unit cross section.

On the other hand, the equation of state defines thermodynamically the nature and the character of the fluid with the following relation that links the pressure, volume and temperature of the fluid:

$$P = f(T, c) \quad (9)$$

For the case of unsaturated soils assuming adiabatic conditions and immiscibility between water and air, the equation of state reduces to the fact that the pressure should be a function of the volumetric water content.

#### *Derivation in Cartesian Coordinates*

Consider a cubical volume of the water conducting soil having its principal axes parallel in direction with the axes of coordinates  $x, y$  and  $z$ . The volume element can be taken very small with sides  $dx, dy$  and  $dz$  oriented in respective directions as shown on figure 1.

Considering now that the soil is homogeneous (having at all points the same macroscopic porosity) but anisotropic with respect to the hydraulic conductivity, the water content and therefore the value of hydraulic conductivity will vary from point to point being thus a single valued function of the coordinates during an infinitesimal time interval. Thus any specific value of hydraulic conductivity can be analyzed into three components  $K_x, K_y$  and  $K_z$  in the directions of the three principal rectangular coordinates.

Since the hydraulic conductivity is a symmetric tensor, the directions of the macroscopic potential gradient do not coincide with the resulting macroscopic flow velocity except at the directions of the principal axes of the ellipsoid. Therefore, if the principal axes are considered parallel to those of the three coordinate axes, application of the dynamic equation of motion

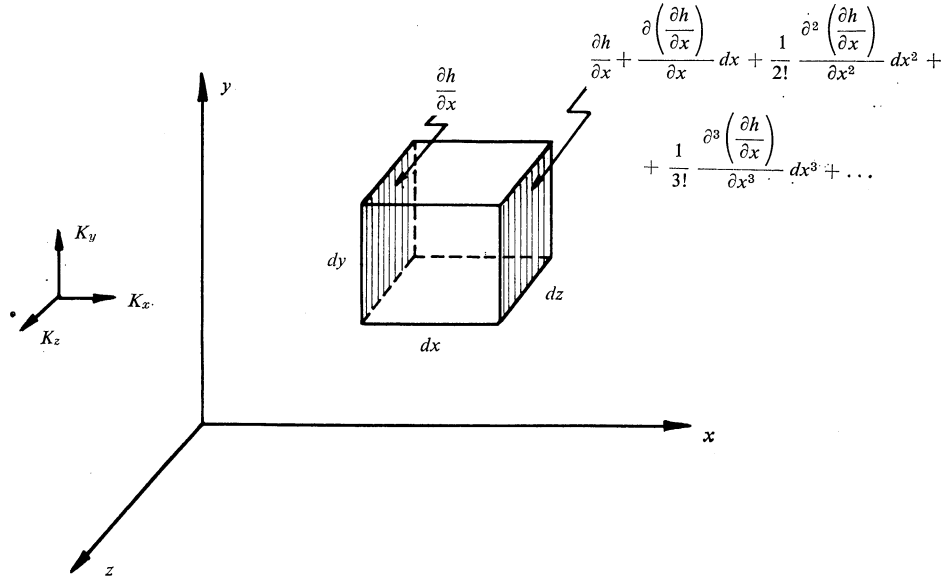


Fig. 1 — A cubical volume element of soil showing the hydraulic gradients imposed on its faces

will render the rate of flow of water through the volume element. Assuming laminar flow throughout the region the dynamic equation of motion is Darcy's law (Eq. 6) specifying the dynamical reactions of the fluid to hydraulic gradient, in the form:

$$\vec{v} = K \nabla h = K \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right) \quad (10)$$

where the minus sign can be omitted with the understanding that the direction of flow is towards decreasing values of the hydraulic gradient.

An applied hydraulic gradient  $\nabla h$  at the  $dy, dz$  face of the element can be resolved into three components  $\partial h/\partial x$ ,  $\partial h/\partial y$  and  $\partial h/\partial z$  in the direction of the three coordinate axes. From the given hydraulic gradient, by using a Taylor series expansion the potential at any opposite face can be found. Expanding therefore, hydraulic gradient in  $x$  direction the gradient on the cubical face opposite to the face  $dy, dz$  is:

$$\frac{\partial h}{\partial x} + \frac{\partial \left( \frac{\partial h}{\partial x} \right)}{\partial x} dx + \frac{1}{2!} \frac{\partial^2 \left( \frac{\partial h}{\partial x} \right)}{\partial x^2} dx^2 + \frac{1}{3!} \frac{\partial^3 \left( \frac{\partial h}{\partial x} \right)}{\partial x^3} dx^3 + \dots \quad (11)$$

By designating the terms after the third one as high order terms expression (11) becomes:

$$\frac{\partial h}{\partial x} + \frac{\partial \left( \frac{\partial h}{\partial x} \right)}{\partial x} dx + 0 (\Delta x^2)$$

The velocity of flow at the six faces of the element is that of hydraulic conductivity times the corresponding hydraulic gradient. Neglecting always the high order terms, since they become continuously smaller as  $\partial h/\partial x$  is made smaller, the obtained results are as shown in figure 2.

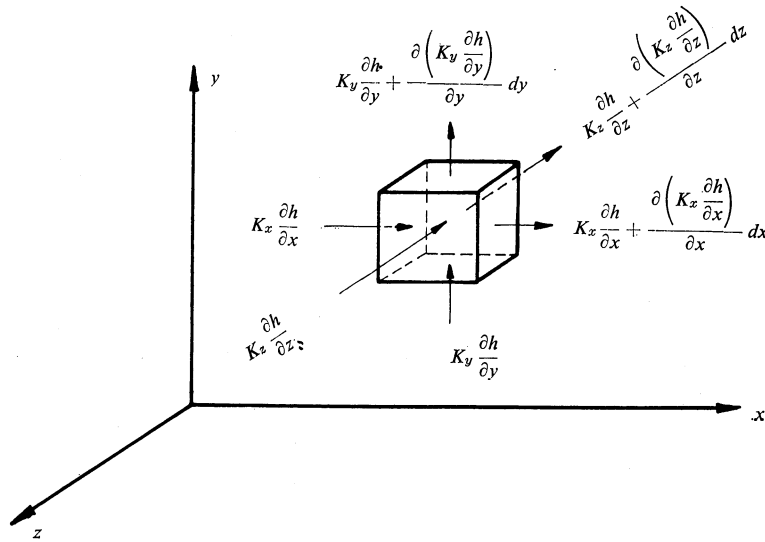


Fig. 2 — The cubical element of soil with the velocities of the flowing water shown on its six faces

The rate of flow of water entering or leaving a face of the cubical element is the product of the velocity and the area of the face. Therefore, for the  $x$  direction the rate of flow of entering the left face is:

$$\left( K_x \frac{\partial h}{\partial x} \right) dy \cdot dz$$

and the rate of flow of water leaving the right face:

$$\left[ K_x \frac{\partial h}{\partial x} + \frac{\partial \left( K_x \frac{\partial h}{\partial x} \right)}{\partial x} dx \right] dy \cdot dz$$

Since the continuity is to be retained the excess flow entering and leaving the cubical element must be equal to the amount stored in the element. Therefore the rate of storage along the  $x$  direction is:

$$\Delta Q_x = \left( K_x \frac{\partial h}{\partial x} \right) dy \cdot dz - \left[ K_x \frac{\partial h}{\partial x} + \frac{\partial \left( K_x \frac{\partial h}{\partial x} \right)}{\partial x} dx \right] dy \cdot dz = - \frac{\partial \left( K_x \frac{\partial h}{\partial x} \right)}{\partial x} dx \cdot dy \cdot dz$$

With a similar procedure the rate of storage along the  $y$  and  $z$  direction is obtained as:

$$\Delta Q_y = \frac{\partial \left( K_y \frac{\partial h}{\partial y} \right)}{\partial y} dy \cdot dx \cdot dz, \quad \nabla Q_z = \frac{\partial \left( K_z \frac{\partial h}{\partial z} \right)}{\partial z} dz \cdot dx \cdot dy$$

The total rate of storage of water in the volume element is therefore the sum of the above three partial rates:

$$\Delta Q = \Delta Q_x + \Delta Q_y + \Delta Q_z = \left[ \frac{\partial \left( K_x \frac{\partial h}{\partial x} \right)}{\partial x} + \frac{\partial \left( K_y \frac{\partial h}{\partial y} \right)}{\partial y} + \frac{\partial \left( K_z \frac{\partial h}{\partial z} \right)}{\partial z} \right] dx \cdot dy \cdot dz \quad (12)$$

Considering now the equation of continuity (8), the rate of storage of water in the element of volume  $dx \cdot dy \cdot dz$  is:

$$\frac{\partial c}{\partial t} dx \cdot dy \cdot dz$$

The above rate is identical with that of expression (12) and therefore equating and dividing by the volume  $dx \cdot dy \cdot dz$  the following expression is obtained:

$$\frac{\partial c}{\partial t} = \nabla K \cdot \nabla h = \frac{\partial \left( K_x \frac{\partial h}{\partial x} \right)}{\partial x} + \frac{\partial \left( K_y \frac{\partial h}{\partial y} \right)}{\partial y} + \frac{\partial \left( K_z \frac{\partial h}{\partial z} \right)}{\partial z} \quad (13)$$

which is the general form of the equation governing the unsaturated flow. Substituting the hydraulic head:

$$h = \frac{p}{\rho g} + z = P + z$$

equation (13) becomes:

$$\frac{\partial c}{\partial t} = K_x \frac{\partial^2 P}{\partial x^2} + \frac{\partial K_x}{\partial x} \frac{\partial P}{\partial x} + K_y \frac{\partial^2 P}{\partial y^2} + \frac{\partial K_y}{\partial y} \frac{\partial P}{\partial y} + K_z \frac{\partial^2 P}{\partial z^2} + \frac{\partial K_z}{\partial z} \frac{\partial P}{\partial z} + \frac{\partial K_z}{\partial z} \quad (14)$$

Also, according to the equation of state (9),  $P=f(c)$  and substituting the diffusion coefficient:

$$D = K \frac{\partial P}{\partial c}$$

equation (13) becomes:

$$\frac{\partial c}{\partial t} = \frac{\partial \left( D_x \frac{\partial c}{\partial x} \right)}{\partial x} + \frac{\partial \left( D_y \frac{\partial c}{\partial y} \right)}{\partial y} + \frac{\partial K_z}{\partial z} \quad (15)$$

The volumetric water content distribution with time can therefore be obtained by any of the expressions (13), (14) and (15) in terms of hydraulic head, pressure or diffusion coefficient.

#### *Derivation in cylindrical coordinates*

The porous medium is assumed again to be anisotropic, homogeneous, unsaturated and extending from  $r_0$  to infinity along any radius  $r$ . Using the cylindrical coordinates  $r$ ,  $\theta$  and  $z$ ,

a volume element of the fluid conducting porous medium has to be taken as the sector  $d\theta$  with thickness  $dz$  showed in figure 3.

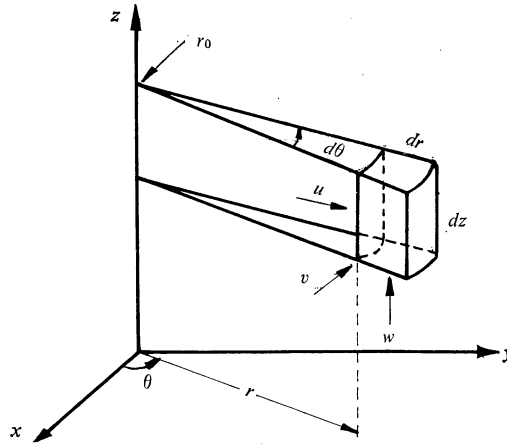


Fig. 3 — Radial sector of soil with the velocity components  $u$ ,  $v$  and  $w$

The above volume element may be considered very small with sides  $dr$ ,  $r d\theta$ ,  $dz$  and  $(r + dr) d\theta$  and with the three velocity components as: the radial  $u$ , the tangential  $v$  and the axial  $w$ . If we denote by  $c$  the volumetric fluid content, the problem is to find a function  $c(r, \theta, t)$  which describes the flow of the water through the unsaturated radial sector of soil.

Taking a section on the  $xy$  plane, the velocity components  $u$  and  $v$  will be:

$$u = K_u \frac{\partial h}{\partial r}$$

and

$$v = K_v \frac{\partial h}{\partial \theta} \quad (16)$$

Using a Taylor series expansion, the corresponding velocity in the  $r$  direction (Fig. 4) will be:

$$u + \frac{\partial u}{\partial r} dr + \frac{1}{2!} \frac{\partial^2 u}{\partial r^2} dr^2 + \frac{1}{3!} \frac{\partial^3 u}{\partial r^3} dr^3 + \dots \quad (17)$$

and the corresponding velocity in the direction of the angle  $\theta$  will be:

$$v + \frac{\partial v}{\partial \theta} d\theta + \frac{1}{2!} \frac{\partial^2 v}{\partial \theta^2} d\theta^2 + \frac{1}{3!} \frac{\partial^3 v}{\partial \theta^3} d\theta^3 + \dots \quad (18)$$

By dropping the high order terms in expressions (17) and (18) and considering the rate of flow entering the element along the  $u$  direction as:

$$u (rd\theta dz)$$

the rate of flow leaving the element will be:

$$\left( u + \frac{\partial u}{\partial r} dr \right) (r + dr) d\theta dz$$

and therefore, the rate of storage along the  $u$  direction is:

$$u (r d\theta dz) - \left( u + \frac{\partial u}{\partial r} dr \right) (r + dr) d\theta . dz$$

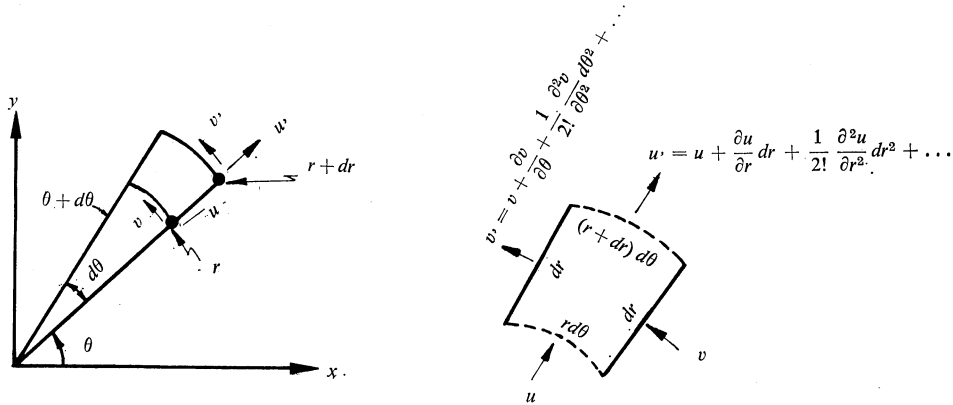


Fig. 4 — Section  $xy$  of the radial sector showing the velocity components

In a similar way the rate of storage along the  $v$  direction is obtained:

$$v (dr dz) - \left( v + \frac{\partial v}{\partial \theta} d\theta \right) dr . dz$$

Therefore, the total rate of storage is:

$$\frac{\partial c}{\partial t} dr d\theta dz = u (r d\theta dz) - \left( u + \frac{\partial u}{\partial r} dr \right) (r + dr) d\theta dz + v (dr dz) - \left( v + \frac{\partial v}{\partial \theta} d\theta \right) dr dz$$

Dividing by  $r dr . d\theta dz$  and simplifying:

$$\frac{\partial c}{\partial t} = u + \frac{\partial u}{\partial r} r + \frac{\partial u}{\partial r} dr + \frac{\partial v}{\partial \theta}$$

Substituting equation (16) and rearranging the final expression is obtained:

$$\frac{\partial c}{\partial t} = \frac{\partial u}{\partial r} + \frac{1}{r} u + \frac{1}{r} \frac{\partial v}{\partial \theta} = \frac{\partial \left( K_u \frac{\partial h}{\partial u} \right)}{\partial r} + \frac{1}{v} \left( K_u \frac{\partial h}{\partial u} \right) + \frac{1}{r} \frac{\partial \left( K_v \frac{\partial h}{\partial v} \right)}{\partial \theta} \quad (19)$$

which is the general form of the unsaturated flow equation in cylindrical coordinates. Any one of equations 13, 14, 15 and equation 19 has a solution for time  $t > 0$  since they are typical equations of a non-reversible process. The solution is uniquely determined if two relationships are defined together with the specified state of the system at the initial time  $t = 0$  and at the boundaries. The two required relations are those of pressure (or head or diffusion coefficient) versus hydraulic conductivity and pressure versus volumetric water content. The necessary requirement is that the two above relations should have no discontinuities since the solution depends continuously on the functions prescribed on the boundaries.

Because of the strong non-linearity in the terms of the equations, they can only be solved by using finite-difference methods. Since the expected numerical operations are great, the use of a desk calculator is excluded and a choice between analogue and digital computer should be made.

By using an analogue computer the corresponding physical problem of flow through an unsaturated porous medium should be replaced by another definite physical process which is more convenient for the purpose of measurement. The relation between these two physical problems is that both should be describable and governed by the same mathematical relationships. The quantities should be continuous and the variables (i.e. time) are represented on certain scales by the machine variables which are voltages, angles or rotation, currents, lengths etc.

On the other hand, a digital computer is not constructed on the principle of an analogy between two processes, but carries out arithmetical and logical operations upon discrete sequences of numbers. It is obvious therefore that a digital computer is suitable for the solution of equations 13, 14, 15 and 19 which have a complete mathematical description and can be solved by numerical methods. Also, since analogue computers depend upon continuous measurements, their accuracy is limited to less than 0.01 per cent whereas there is no such limit in theory to the accuracy of a digital computer.

Therefore, for any desired solution a digital computer should be used and the problem is that of approximating the differentials with finite differences at discrete points in the solution domain. An analysis of the truncation error due to the neglect of the high-order terms in the Taylor's series expansion and the propagation and growth of the round-off error due to the rounding off procedure in the repeated application of the recurrence scheme, is also necessary.

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