An analytical solution for land subsidence in the clayey layer due to pumping

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Abstract Serious land subsidence occurs in the western and southwestern coastal regions of Taiwan due to over-pumping of groundwater from the aquaculture industry. Many coastal aquifers in Taiwan are composed of multi-layered aquifers which include many layers of sandy aquifers and semi-impervious layers (i.e., aquitards). Changes in groundwater levels in sandy aquifers following pumping may produce consolidation in aquitards. Previously, a soil one-dimensional (1D) consolidation model caused by the change in hydraulic head was solved by a finite difference method with considering viscosity effect. In this study, the solution in Laplace domain for the force balance equation coupled with the flow equation is first developed using the Laplace transforms. The time domain results are then obtained by means of a numerical inversion scheme. The solution can be used as a preliminary assessment tool in coastal aquifer management, where it is necessary to estimate the consolidation and excess pore water pressure of clay stratum as a result of pumping in sandy aquifers.

Keywords land subsidence; excess pore water pressure; aquitards; coastal aquifers; Laplace transforms

INTRODUCTION

Serious land subsidence occurs in the western and southwestern coastal regions of Taiwan due to over-pumping of groundwater from the aquaculture industry. Land subsidence produces the cause of land flooding from heavy rainfall and tide overflow from the sea during typhoon seasons and often results in heavy damage of public facilities and private properties. The problems of land subsidence due to over-pumping of groundwater become active research areas within recent one or two decades. Hence, the analyses and predictions of soil consolidation due to groundwater withdrawal are necessary to prevent land subsidence and serve as constraints in the sustainable use of groundwater resource needs.

Lewis & Schrefler (1978) presented a fully coupled finite element consolidation model to simulate the subsidence and the piezometric decline in Venice, Italy. Yeh *et al*. (1996) developed a finite element model based on the combination of three dimensional equations of force balance and saturated/unsaturated flow to simulate the land displacements due to pumping. Kihm *et al*. (2007) simulated numerically a three-dimensional fully coupled groundwater flow and land deformation due to groundwater pumping in an unsaturated fluvial aquifer system. Wu *et al*. (2010) developed a coupled model consisting of a three-dimensional groundwater flow model and a one-dimensional (1D) vertical deformation model, both based on a viscoelastoplastic constitutive law, to simulate the land subsidence occurring in Shanghai, China. Ortiz-Zamora and Ortega-Guerrero (2010) performed both field investigations and predictive simulations based on a one-dimensional, nonlinear groundwater flow-consolidation model to study the evolution of long-term land subsidence near Mexico City.

Tsai (2009) presented a 1D mathematical model for consolidation of poroelastic soil based on the equation for forces equilibrium in incremental state coupled with the saturated flow equation. The finite difference method along with the Thomas algorithm was then employed to solve the coupled equations for the excess pore water pressure and displacement in the clay formation. This study is first to develop the solution in Laplace domain based on Tsai's consolidation model (2009) using the Laplace transforms. The time domain results are then obtained by means of a numerical inversion scheme. In addition, a closed-form solution in time domain is also developed for a special case. These solutions can be used as a preliminary assessment tool in coastal aquifer management, where it is necessary to estimate the consolidation and excess pore water pressure of clay stratum as a result of pumping in sandy aquifers. In addition, these solutions are useful in practical applications because they can also be used to evaluate the sensitivities of the model parameters, to verify a numerical code, or to check the accuracy of the numerical results.

THE MATHEMATICAL MODEL

The flow equation for describing 1D soil consolidation can be expressed as (Tsai 2009)

$$\frac{\partial}{\partial z}\left[K \ \frac{\partial P^e}{\partial z}\right] = \rho_w g \frac{\partial^2 u_z}{\partial t \ \partial z} \tag{1}$$

where z is the coordinate, K is the hydraulic conductivity, P^e is the excess pore water pressure, ρ_w is the water density, t is the time, u_z is the vertical soil displacement. The equation for equilibrium of forces for linear viscoelastic soil can be written as

$$\frac{\partial}{\partial z}\left[(2G+\lambda)\frac{\partial u_z}{\partial z}\right]+\frac{\partial}{\partial z}\left[(2G'+\lambda')\frac{\partial^2 u_z}{\partial t\,\partial z}\right]=\frac{\partial P^e}{\partial z} \tag{2}$$

where G and λ are Lame's constant and G' and λ' are viscous moduli.

The initial excess pore water pressure and soil displacement are considered as zero, i.e.,

$$P^e(z,0)=0 \tag{3}$$

and

$$u_z(z,0)=0 \tag{4}$$

The excess pore water pressures at the boundaries of the bottom and top of clay layer are proportional to the changes in head following pumping in the overlain and underlain sand formations and thus, respectively, expressed as

$$P^e(0,t)=-\rho_w g h_2(t) \tag{5}$$

and

$$P^e(B,t)=-\rho_w g h_1(t) \tag{6}$$

where B is the thickness of clay layer. Assume that the displacement at the bottom of clay is equal to zero. That is

$$u_z(0,t)=0 \tag{7}$$

Consider that the upper sandy aquifer is under confined condition as illustrated in Fig. 1. An increment in effective stress is produced at the top of the clay, i.e., $z=B$, by the drawdown as a result of pumping. Thus, the top boundary condition for the clay layer may be written as

$$(2G+\lambda)\frac{\partial u_z(B,t)}{\partial z}+(2G'+\lambda)'\,\frac{\partial^2 u_z(B,t\ )}{\partial z\,\partial t}=-\rho_w g h_1\ (t) \tag{8}$$

Tsai (2009) defined the reference soil displacement and reference time for the hydraulic conductivity K_f and Lame's constants G_f and λ_f as $\rho_w gB^2/(2G_f+\lambda_f)$ and $\rho_w gB^2/(2G_f+\lambda_f)K_f$, respectively. For brevity of presentation, following dimensionless variables and parameters are introduced: $P^{e*}=P^e/\rho_w gB$, $u_z^*=\lambda_f u_z/\rho_w gB^2$, $t^*=K_f\lambda_f t/\rho_w gB^2$, $z^*=z/B$, $N=\lambda_f' K_f/\rho_w gB^2$, $K^*=K/K_f$,

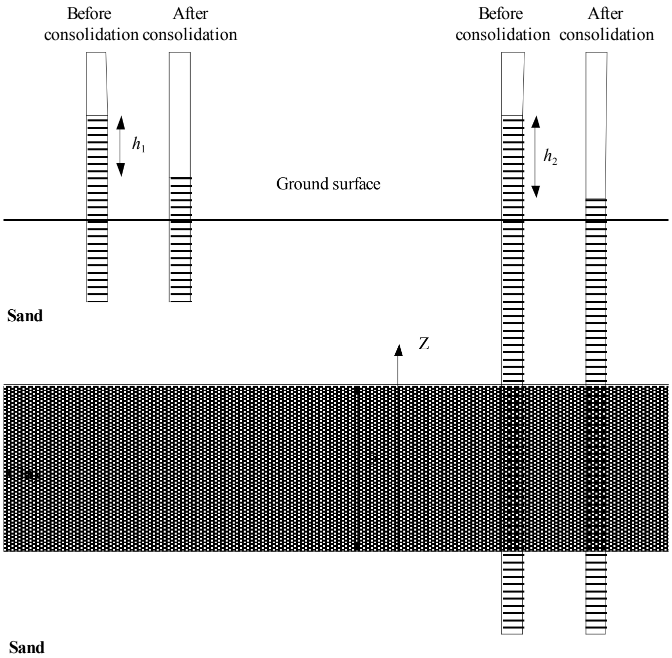


Fig. 1 Schematic diagram of soil consolidation due to the change in groundwater level in response to the pumping in sandy aquifers (modified from Tsai, 2009).

and $\lambda^* = \lambda / \lambda_f$. Accordingly, Eqs. (1) and (2) can be nondimensionalized, respectively, as

$$\frac{\partial}{\partial z^*}\left[K^* \frac{\partial P^{e*}}{\partial z^*}\right] = \frac{\partial^2 u_z^*}{\partial t^* \partial z^*} \tag{9}$$

and

$$\frac{\partial}{\partial z^*}\left[\left(2G^* + \lambda^*\right)\frac{\partial u_z^*}{\partial z^*}\right] + \frac{\partial}{\partial z^*}\left[N \frac{\partial^2 u_z^*}{\partial t^* \partial z^*}\right] = \frac{\partial P^{e*}}{\partial z^*} \tag{10}$$

where N is viscosity number. The boundary conditions for the excess pore water pressures at $z^* = 0$ and $z^* = 1$ are, respectively,

$$P^{e*}\left(0, t^*\right) = -\frac{h_2\left(t^*\right)}{B} \tag{11}$$

and

$$P^{e*}\left(1, t^*\right) = -\frac{h_1\left(t^*\right)}{B} \tag{12}$$

Furthermore, the boundary conditions, Eqs. (7) and (8), for the soil displacement may be nondimensionalized, respectively, as

$$u_z^*\left(0, t^*\right) = 0 \tag{13}$$

and

$$\left(2G^* + \lambda^*\right)\frac{\partial u_z^*\left(1,t^*\right)}{\partial z^*} + N\frac{\partial^2 u_z^*\left(1,t^*\right)}{\partial z^* \partial t^*} = -\frac{h_1\left(t^*\right)}{B} \tag{14}$$

Additionally, the initial conditions for the excess pore water pressure and soil displacement can be written, respectively, as

$$P^{e*}(z^*,0) = 0 \tag{15}$$

and

$$u_z^*(z^*,0) = 0 \tag{16}$$

Applying the Laplace-transform to Eqs. (1) – (16), the solutions for soil displacement and excess pore water pressure in the Laplace-domain can be obtained, respectively, as

$$\widetilde{P}^{e*} = -\frac{e^{\sqrt{\omega}z^*} - e^{-\sqrt{\omega}z^*}}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}}\frac{\widetilde{h}_1(s)}{B} - \frac{e^{\sqrt{\omega}\left(1-z^*\right)} - e^{-\sqrt{\omega}\left(1-z^*\right)}}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}}\frac{\widetilde{h}_2(s)}{B} \tag{17}$$

and

$$\widetilde{u}_z^* = \frac{K^*\sqrt{\omega}}{s}\left[\frac{2}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}} - \frac{e^{\sqrt{\omega}z^*} + e^{-\sqrt{\omega}z^*}}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}}\right]\frac{\widetilde{h}_1(s)}{B} + \frac{K^*\sqrt{\omega}}{s}\left[-\frac{e^{\sqrt{\omega}} + e^{-\sqrt{\omega}}}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}} + \frac{e^{\sqrt{\omega}\left(1-z^*\right)} + e^{-\sqrt{\omega}\left(1-z^*\right)}}{e^{\sqrt{\omega}} - e^{-\sqrt{\omega}}}\right]\frac{\widetilde{h}_2(s)}{B} \tag{18}$$

where s is the Laplace variable and

$$\omega = \frac{s}{K^*\left[(2G^* + \lambda^*) + Ns\right]} \tag{19}$$

Special case

Consider a special case of negligible viscosity effect that $G^* = N = 0$, $K^* = \lambda^* = 1$, $h_1 = 0$, and $h_2/B = 0.01$. The parameter in Eq. (19) becomes $\omega = s$ and Eqs. (17) & (18) can, respectively, reduce to:

$$\widetilde{P}^{e*} = -\frac{e^{\sqrt{s}\left(1-z^*\right)} - e^{-\sqrt{s}\left(1-z^*\right)}}{e^{\sqrt{s}} - e^{-\sqrt{s}}}\frac{0.01}{s} \tag{20}$$

and

$$\widetilde{u}_z^* = \frac{1}{\sqrt{s}}\left[-\frac{e^{\sqrt{s}} + e^{-\sqrt{s}}}{e^{\sqrt{s}} - e^{-\sqrt{s}}} + \frac{e^{\sqrt{s}\left(1-z^*\right)} + e^{-\sqrt{s}\left(1-z^*\right)}}{e^{\sqrt{s}} - e^{-\sqrt{s}}}\right]\frac{0.01}{s} \tag{21}$$

The Laplace inversions of Eqs. (20) & (21) are, respectively,

$$P^{e*}\left(z^*,t^*\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\widetilde{P}^{e*}\left(z^*;s\right)e^{st}ds \tag{22}$$

and

$$u_z^*\left(z^*,t^*\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\widetilde{u}_z^*\left(z^*;s\right)e^{st}ds \tag{23}$$

Therefore, the excess pore water pressure and soil displacement in the time domain can then be, respectively, obtained as

$$P^{e*}\left(z^*,t^*\right)=-0.01\left\{\left(1-z^*\right)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}e^{-n^2\pi^2t^*}\sin\left[\left(1-z^*\right)n\pi\right]\right\} \tag{24}$$

and

$$u_z^*\left(z^*,t^*\right)=\frac{0.02}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^2}\left(1-e^{-n^2\pi^2t^*}\right)\left\{\cos\left[\left(1-z^*\right)n\pi\right]-\left(-1\right)^n\right\} \tag{25}$$

RESULTS AND DISCUSSION

Fig. 2 shows the simulated results for the excess pore water pressure curves and soil displacement curves by Eqs. (24) & (25), respectively, for the dimensionless time at 0.05, 0.1, 0.3, and steady-state for $G^*=N=0$, $K^*=\lambda^*=1$, h_2 /B=0.01, and h_1=0. This figure gives the excess pore water pressure and soil displacement distributions in a homogeneous soil without considering the viscosity effect. The curves shown in the figure perfectly match with those curves given in Tsai (2009, Fig. 3), indicating that the correctness of Tsai's numerical results for the case without the viscosity effect. Note that there is a typo of using $G^*=1$ in Tsai (2009) which should read as $G^*=0$. To examine the viscosity effect on the consolidation of poroelastic soil, Tsai (2009) used a case of a homogeneous soil for $K^*=G^*=\lambda^*=1$ and h_1 /B=h_2 /B=0.01. This case is also adopted to predict the soil displacement versus the dimensionless vertical depth at various dimensionless times when N = 0.001, 0.01, and 0.1. The numerical inversion scheme, Stehfest algorithm (1970), is used to invert Eq. (18) for the soil displacement and the results are shown in Fig. 3. This figure shows that the soil with a larger N, the viscosity number, will have a smaller displacement. Obviously, the viscosity effect plays an important role in consolidation for the clay layer under the condition of large pumping in adjacent aquifers. Again, these results obtained from the Laplace-domain solution match very well with the finite-difference solutions as shown in Tsai (2009, Fig. 5).

CONCLUDING REMARKS

This study has developed the solution in Laplace domain for the predictions of excess pore water pressure and soil displacement based on Tsai's 1D consolidation model with considering viscosity effect for poroelastic soil due to the change in head from the adjacent aquifers. The time-domain results can then be easily obtained via Stehfest

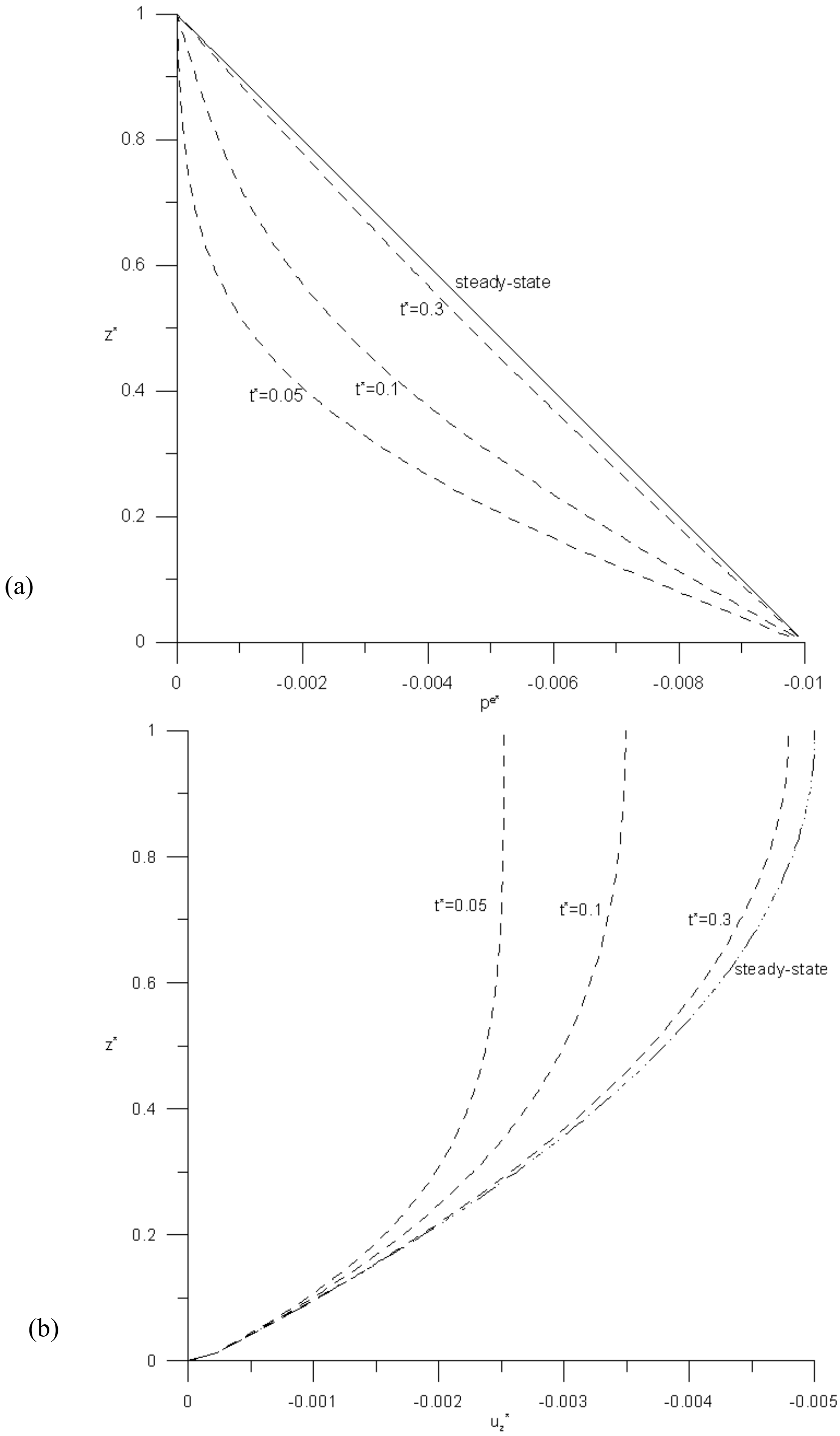
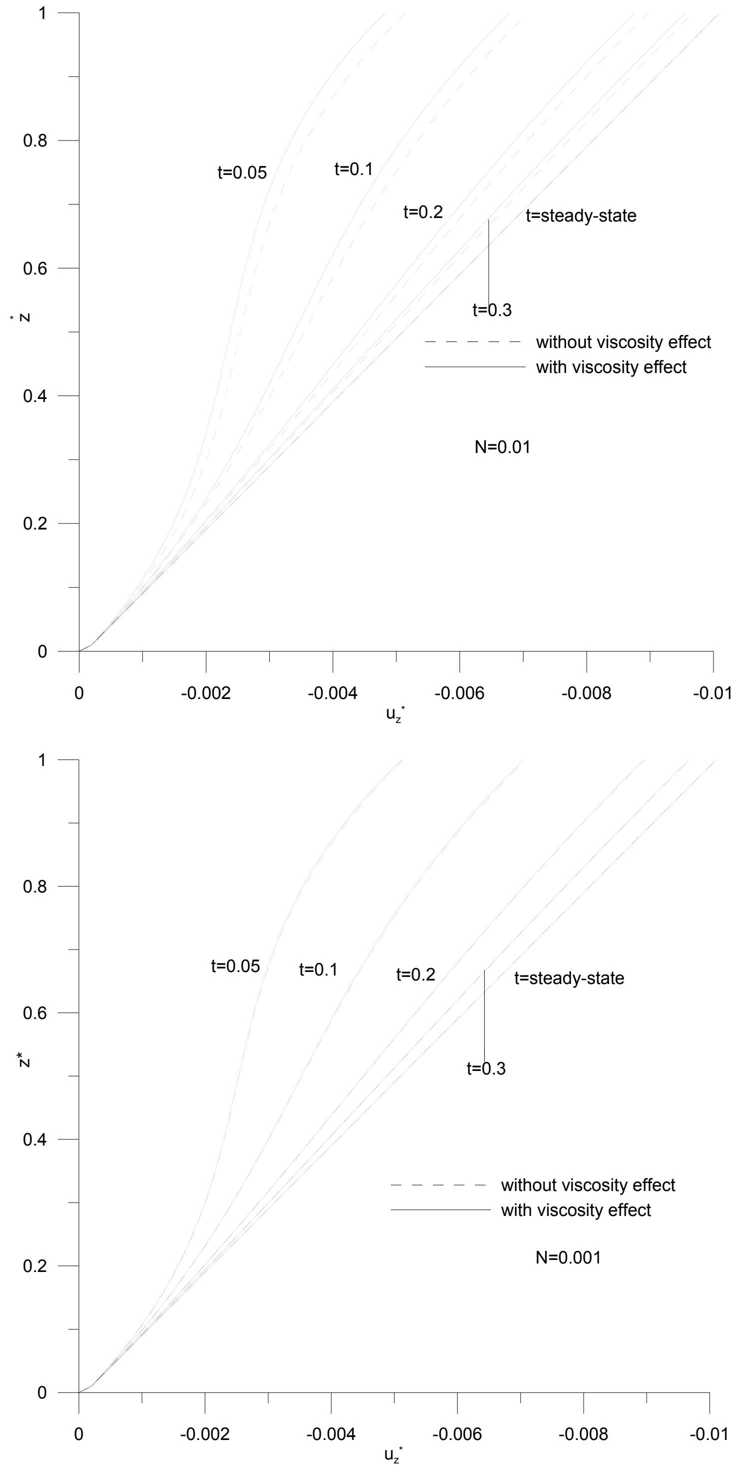


Fig. 2 The simulated results for (a) excess pore water pressure and (b) soil displacement at different dimensionless time for G^*=N=0, K^*=λ^*=1, h_2/B=0.01 and h_1=0..

algorithm. In addition, an analytical solution is also derived using the inverse Laplace transforms for the special case of negligible viscosity effect. The predicted results from both the Laplace-domain solution and the analytical solution for the special case show excellent agreement with those from the finite different solutions. In addition, the predicted results from the Laplace-domain solution for the displacement indicate that

(a)

(b)



the viscosity effect is crucial and should be consider for consolidation in the clay layer when the adjacent aquifers are subject to a large pumping rate.

(c)

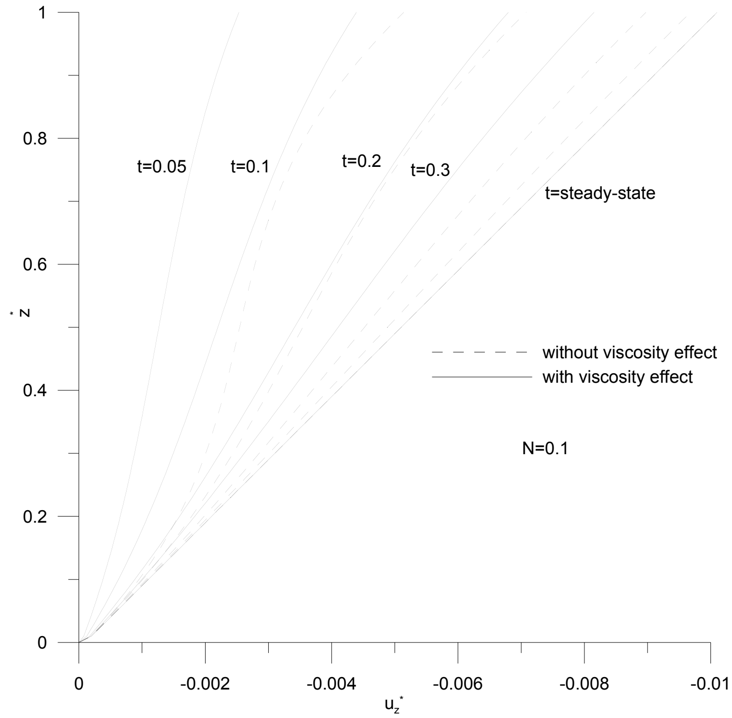


Fig. 3 The predicted soil displacements versus dimensionless vertical depth for $K^{*}=G^{*}=\lambda^{*}=1$ and $h_1/B=h_2/B=0.01$ at various dimensionless times when $N = 0.001$, 0.01, and 0.1.

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