THE FLOW LAW OF ICE
A discussion of the assumptions made in glacier theory, their experimental foundations and consequences

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SUMMARY

Experimental evidence on the flow law of ice is reviewed, and the justification for various assumptions commonly made in theoretical studies of ice movement is discussed. This enables the reliability of results obtained using the assumptions to be assessed.

The general theory developed allows certain predictions to be made concerning the effects of complicated stressing systems, and in particular the theory is applied qualitatively to explain the anomalous behaviour of glaciers below ice falls and in other places where large stresses are acting in the ice due to its flow.

Introduction

Theoretical discussions of the physics of ice movement must start from some assumptions concerning the behaviour of ice when it is subjected to a stress. In this paper we shall consider the experimental evidence on which such assumptions can be based, and so bring out the possible reasons for which the theory could fail. In this way it is possible, to assess the strength and weakness of the present state of the theory of ice movement, and to see in what directions future investigations should proceed. It is also possible to see certain predictions of a theoretical nature which may assist in the explanation of some current glacier problems.

Experimental evidence on the flow law of ice

Experimental evidence on the flow law of ice can be gained in two main ways, from laboratory tests, and from the success of theoretical predictions of the behaviour of natural ice masses. In many ways the simplest method consists of laboratory work, for the conditions of the experiment are usually much better defined than in field work, and, although the success of theoretical work on glacier flow is a heartening confirmation of the assumptions made, laboratory tests provide the best foundation for our knowledge of the behaviour of ice.

A large number of tests have been performed on the mechnanical behaviour of ice under various kinds of stress. Many of the more modern ones have recently been reviewed (1), and I shall simply summarize the position here. Single crystals of ice can deform by slip on their basal planes, and no definite stress has been found below which no deformation occurs. Under a given stress the rate of deformation appears to increase with time, irrespective of whether the stress is applied as a compression, tension or shear (2) (3) (4). There is some doubt concerning the mathematical form of the creep curve, and there is as yet little evidence on the variation of flow with stress, although it would seem to be something rather like a power law the exponent of which lies between 2 and 4. The variation with temperature is quite marked, and one relation which has been dedeuced (2) is that as the temperature is reduced from 0°C, the stress required to produce a given creep curve is increased by a factor proportional to the square of the temperature in degrees Centigrade. It has also been shown (4) that prolonged resting, or changing the glide direction, does not restore its original hardness to an ice crystal, but that, once it has been deformed and the creep has accelerated, this accelerated rate is resumed if the stress is reimposed. A hydrostatic pressure does not affect the creep behaviour of an ice crystal, provided the difference between the test temperature and the melting point is kept constant (5). However it has been reported (6) that a compressive stress perpendicular to the slip plane increases the glide rate considerably.

Tests on randomly oriented polycrystalline ice (7) (8) have shown that, under a constant load, the rate of flow decreases with time at first, though it may later settle down to a steady rate or even reaccelerate. This rather surprising difference from the behaviour of single crystals must be due to interaction between the various grains, while the subsequent reacceleration can be attributed to the recrystallisation of the ice with a preferred orientation favorable to slip under the action of the applied stresses, for this does appear to be the preferred orientation observed. The minimum flow rate observed can be related to the applied stress, and it is found that a power law of the form

$$e = (\sigma/A)^n \tag{1}$$

results, where e is the strain rate, σ is the applied stress, and A and n are constants. The value found for n in tests at higher stresses was about 4, but at lower stresses the value of n appeared to drop; however this may be due solely to the fact that the tests at low stresses were not continued long enough for the true minimum rate to be observed, and when this is taken into account, no important deviations from this law were observed. This law also results from certain analyses of the flow of glacier ice, but since the stresses are not simple in these cases, the considerations discussed later in this paper have to be borne in mind in assessing the results. The variation of flow with temperature has also been investigated, but not over a sufficiently large range of temperatures to enable the law governing it to be deduced. The magnitude of the variation can be quite large: about a factor 6 in flow rate between the melting point and — 13°C.

Most of the tests on polycrystalline ice in which the stress has been uniform throughout the specimen have been performed in tension or compression (though recently a series of shear tests has been reported (9)), but it is most important to see what the effect of more complex stress systems is, and it is worth emphasizing that there is no general theory by which the behaviour under complex stress systems can be deduced completely from the results of tests in simple tension or compression without making drastic assumptions the nature of which will be discussed below.

Steinemann (9) has investigated the effect on shear tests of a large hydrostatic pressure, and has found that there is no marked effect, and has also studied the effect of superimposing uniaxial compression and shear. In this latter experiment the specimen was ring shaped, and was twisted to give the shear, the compression being applied axially. This experiment is virtually the only one that has been reported in which different stressing systems have been applied to essentially similar specimens, and the results are thus of great importance for the testing of any theory of ice flow; they will be discussed again below.

The experimental information is, as we see, still rather sparse, but we can consider what the theoretical possibilities are, and can then investigate what such evidence as we have can tell us concerning the various possibilities. Finally we can see what consequences in terms of the behaviour of ice in nature would result if certain theories were to prove to be applicable.

Theoretical possibilities for the flow law of ice

In order to limit this discussion, I will assume that the ice we are considering consisted originally of randomly oriented polycrystals, and was thus isotropic. This assumption is seldom true for natural ice masses, but is probably a good approximation in the case of glacier ice, though it would be a rather poor one for some other forms such as river ice. I will also assume that the rate of flow depends only on the stress, and not upon the time for which it has been acting. We may hope that this will be true for those cases of flow where a particular piece of ice remains under the same stress sufficiently long for a steady state to be reached. It may be that recrystallisation will already have produced ice far from isotropic before this steady state is reached, but, as is shown in the Appendix, provided the stress does not change direction, this anisotropy does not invalidate the theory. These initial assumptions naturally impose limitations on the theory, and if any particular results do not agree with theoretical predictions, it may always be because these initial assumptions are untrue.

Even in an initially isotropic material in which the flow is a function of the stress alone, the flow law can be quite complex. It can be shown on general grounds (see Appendix) that the relation between two second rank tensors such as the strain rate e and the stress σ must be of the form

$$e_{ij} = A(\Sigma_1, \Sigma_2, \Sigma_3,)\, \delta_{ij} + B(\Sigma_1, \Sigma_2, \Sigma_3)\, \sigma_{ij} + C(\Sigma_1, \Sigma_2, \Sigma_3)\, \sigma_{ik}\, \sigma_{kj}, \qquad (2)$$

where the suffices i and j can take any of the values 1, 2 or 3 and so give the various components of the two tensors, δ_{ij} is equal to 1 if $i = j$, but zero if $i \neq j$, and A, B and C are three general functions of the three invariants of the stress tensor Σ_1, Σ_2 and Σ_3. Here, and throughout this paper, we use the summation convention for repeated suffices.

The three invariants of the stress tensor can be written

$$\Sigma_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad \Sigma_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1),$$
$$\Sigma_3 = \sigma_1\sigma_2\sigma_3,$$

where σ_1, σ_2 and σ_3 are the three principal stresses. Σ_1 is just three times the hydrostatic pressure. However one of the few things that the experiments do tell us is that in those cases where it has been tested, the hydrostatic pressure does not affects the flow law, and thus Σ_1 must not enter our equations. A very convenient way of ensuring that this is so is to define the stress deviator by the equation

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\Sigma_1,$$

the three invariants of this tensor are

$$\Sigma'_1 = \sigma'_1 + \sigma'_2 + \sigma'_3 = \sigma'_{ii} = 0,$$
$$\Sigma_2 = -(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1) = \frac{1}{2}\sigma'_{ij}\sigma'_{ij},$$
$$\Sigma'_3 = \sigma'_1\sigma'_2\sigma'_3 = \frac{1}{3}\sigma'_{ij}\sigma'_{jk}\sigma'_{ki}$$

and the first invariant is necessarily zero. We can therefore rewrite equation (2) in terms of the stress deviator; it becomes

$$e_{ij} = A(\Sigma'_2, \Sigma'_3)\, \delta_{ij} + B(\Sigma'_2, \Sigma'_3)\, \sigma'_{ij} + C(\Sigma'_2, \Sigma'_3)\, \sigma'_{ik}\, \sigma'_{kj}. \qquad (3)$$

We can further reduce the possibilities for the flow law by using the fact that the density of ice remains constant during flow; this will be true if the properties of the ice are not changing, and this is the only case for which any general theory can hold. The change in density is related directly to the first invariant of the strain rate tensor $E_1 = e_{ii}$. Tjus if the density does not change, $E_1 = 0$, and if we use equation (3) to determine e_{ii}, and equate the result to zero we get

$$3\,A(\Sigma'_2, \Sigma'_3) + B(\Sigma'_2, \Sigma'_3)\,\Sigma'_1 + C(\Sigma'_2, \Sigma'_3)\,2\,\Sigma'_2 = 0,$$

or, since $\Sigma'_1 = 0$,

$$A(\Sigma'_2, \Sigma'_3) = -\frac{2}{3}\Sigma'_2\,C(\Sigma'_2, \Sigma'_3).$$

This is a relation between A and C which we can use to eliminate A from equation (3) and so obtain

$$e_{ij} = -\frac{2}{3}\Sigma'_2\,C(\Sigma'_2, \Sigma'_3)\,\delta_{ij} + B(\Sigma'_2, \Sigma'_3)\,\sigma'_{ij} + C(\Sigma'_2, \Sigma'_3)\,\sigma'_{ik}\sigma'_{kj}, \quad (4)$$

which is the most general form that the relation between stress and strain rate can have under our assumptions. It contains two arbitrary functions of two variables, and these cannot be determined from any series of tests using only one kind of loading, such as for example compression tests. This demonstrates the fact that such a series of tests does not give sufficient information completely to determine the flow law of ice without further assumptions.

In order to do some calculations based on laboratory tests in compression, Nye ([10]) made the further assumption that under a given stress the components of strain rate are proportional to the components of the stress deviator and that the second invariant of the strain rate tensor is a function of Σ'_2 only, and if this is so, it can be shown (see Appendix) that equation (4) reduces further to

$$e_{ij} = B(\Sigma'_2)\,\sigma'_{ij}. \quad (5)$$

This relation can be tested using the results from any experiments in which, for comparable specimens, Σ'_2 and Σ'_3 were varied separately. Such tests on polycrystals have been made by Steinemann ([9]). It is possible to compare his tests in uniaxial compression and tension with his tests in shear by plotting values of E_2, the second invariant of the strain rate tensor, against values of Σ'_2; if this is done all the points should lie on a single curve. This procedure is equivalent to that adopted by Nye ([10]) in comparing laboratory compression tests with the glacier results, for Nye's τ is equal to the square root of Σ'_2, and Nye's $\dot{\varepsilon}$ is equal to the square root of E_2. Steinemann has made this comparison and finds that the two types of test do not give the same curve, that for shear giving a slower strain rate at a given stress than is predicted by the theory from the uniaxial results. Steinemann has suggested that this may due to a different anisotropy of the ice developing in the two specimens, but this anistropy, being entirely determined by the stress, can only affect B and C in equation (4). It is quite possible to account for Steinemann's result provided the function C in equation (4) is not zero, or B is function of Σ'_3.

Steinemann ([9]) has also made tests under simultaneous shear and compression, as described above, and these tests also allow us to test equation (5). Here the test may be even better than in the comparison between uniaxial and shear tests, as specimens of the same shape were used throughout. The method of making this comparison is discussed in the next section.

Interpretation of Steinemann's tests under simultaneous shear and compression

In these experiments the direction of compression lay in the plane of shear and parallel to one of its directions (see Fig. 1). If we represent the magnitude of the

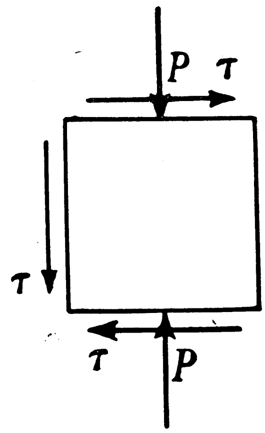

Fig. 1 — Diagram to show the combination of stresses used by Steinemann in his tests under combined shear und compression.

shear stress by τ and that of the uniaxial compression by P, then the stress tensor can be written in matrix form as

$$\begin{pmatrix} 0 & \tau & 0 \\ \tau & P & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the stress deviator is

$$\begin{pmatrix} -P/3 & \tau & 0 \\ \tau & 2P/3 & 0 \\ 0 & 0 & -P/3 \end{pmatrix}.$$

The second invariant of the stress deviator

$$\Sigma'_2 = \frac{1}{3} P^2 + \tau^2,$$

and, if equation (5) holds, E_2 will be a function of Σ'_2 only, and the individual components of strain rate will, according to the assumption from which (5) was derived, be proportional to the individual components of the stress deviator. Steinemann only reports measurements of the shear strain, but the specimen must also have been compressing under the action of the compressive stress. It the resulting shear strain rate is $\dot{\gamma}$ (using the usual definition of shear strain, so that the tensor component $e_{12} = \frac{1}{2} \dot{\gamma}$), and the uniaxial compression strain rate is $\dot{\varepsilon}$, then the strain rate tensor expressed in matrix form will be

$$\begin{pmatrix} -\dot{\varepsilon}/2 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & \dot{\varepsilon} & 0 \\ 0 & 0 & -\dot{\varepsilon}/2 \end{pmatrix},$$

and its second invariant

$$\frac{3}{4} \dot{\varepsilon}^2 + \frac{1}{4} \dot{\gamma}^2,$$

and so it follows form equation (5) that

$$\frac{3}{4}\dot{\varepsilon}^2 + \frac{1}{4}\dot{\gamma}^2 = f\left(\frac{1}{3}P^2 + \tau^2\right)$$

Thus for example when $P = 0$, and, consequently, $\dot{\varepsilon} = 0$,

$$\frac{1}{4}\dot{\gamma}^2 = f(\tau^2).$$

Since Steinemann did not measure $\dot{\varepsilon}$, we cannot check this prediction directly, but we can compute what $\dot{\varepsilon}$ should have been and then use this value to analyse the results. If the components of strain rate and stress deviator are proportional, then

$$2\,\tau/\dot{\gamma} = 2\,P/(3\dot{\varepsilon}),$$

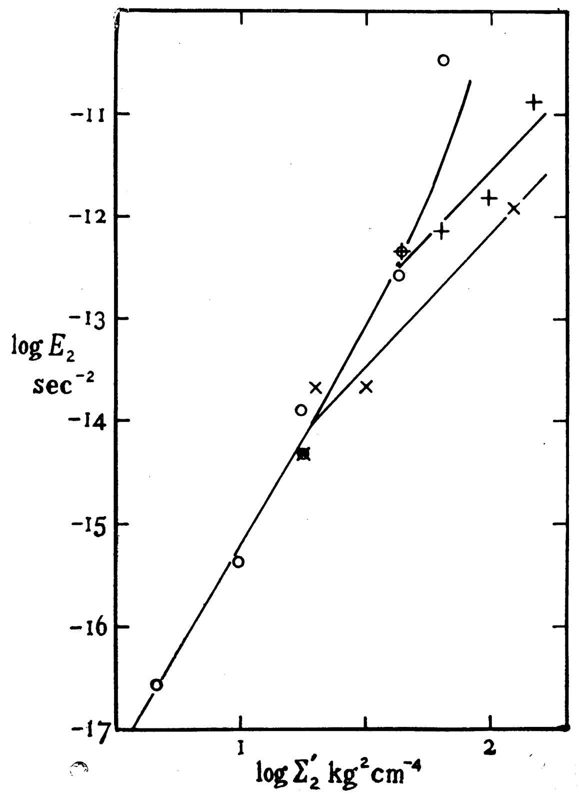

Fig. 2 — A graph of log E_2 against log Σ_2 using the data from Steinemann's tests under combined shear and compression. If the assumptions suggested by Nye were true, all the points should lie on one curve. The symbols used for points have the following meaning:

- O points derived from tests in simple shear
- × points derived from tests under a shear stress of 4.20 Kg./cm². with superimposed compressive stresses from 0 to 18 Kg./cm².
- + points derived from tests under a shear stress of 6.59 Kg./cm². with superimposed compressive stresses from 0 to 18 Kg./cm².

or

$$\dot{\varepsilon} = P\dot{\gamma}/3\tau.$$

Thus if we plot $\frac{1}{4}\,\dot{\gamma}^2(1 + P^2/3\tau^2)$ against $\frac{1}{3}\,P^2 + \tau^2$ we should get the same curve whether or not the compression is acting. In fact since a power law is a good approximation to this relation, it is best to plot the logarithm of these two quantities, and this is what has been done in Fig. 2 using data obtained from a graph of Steinemann's results.

It will be seen from Fig. 2 that, although there is some scatter among the points, as is to be expected in any mechanical tests of this sort, there is a definite difference between points corresponding to a given value of τ, such that the effect of a compressive stress P on the shear rate is not as great as the effect of a shear stress giving the same value of Σ'_2. This is not necessarily in contradiction with the result found in comparing shear and compression tests, where the effect of compression was to give a faster starin rate that that for a comparable shear, becuse in that case it was the compression strain that was measured under the compressive stress. It is quite possible that the value of $\dot{\varepsilon}$ in these tests was correspondingly large.

Applications to the theory of glacier flow

It is useful to see what implications the theory that we have been discussing has concerning the behaviour of glaciers. Nye ([10]) has drawn many fruitful deductions from the law expressed by equation (5), and there is no need to repeat that work; here we shall simply consider briefly what consequences would follow if, as seems now rather probable, we have to admit that equation (5) is not accurate, and we adopt instead equation (4). Finally we can consider a few implications of equation (5) that have not previously been discussed, and which may still have some validity in terms of the more general law.

One of the main differences between equations (4) and (5) is the presence in equation (4) of the terms containing the function C which show that in general e_{ij} will not be zero even if the corresponding stress deviator component σ'_{ij} is zero. For example, a specimen of ice would swell or contract perpendicular to the plane of shear, a behaviour well-known to occur in other non-Newtonian materials. This can be shown as follows. If a simple shear stress, for which the only non-zero term of the stress deviator is σ'_{12}, is applied, then Σ'_2 will be non-zero but Σ'_3 will be zero. The swelling perpendicular to the plane of shear will be given by

$$e_{33} = -\frac{2}{3}\Sigma'_2\, C\,(\Sigma'_2, \Sigma'_3),$$

and so unless $C(\Sigma'_2, \Sigma'_3)$ happens to be zero whenever Σ'_3 is zero, e_{33} will be finite, and the material will grow (or contract) perpendicular to the plane of shear. If it is not free to change dimensions in this way, as would be the case if the ice were part of a valley glacier, then a stress deviator σ'_{33} would have to develop to prevent the change. Experiments which test whether such an expansion occurs, or which measure a cross-stress developed if the expansion is prevented, are therefore a way of investigating the form of the function C.

Such experiments do not seem to have been done for ice, and until they are there remain many possibilities open concerning the form of the function C. There is only one further theoretical limitation, and this is made by the fact that rate of working of the applied stresses must be positive, and so $\sigma'_{ij}\, e_{ij}$ must be positive. If this is substituted in equation (4) an inequality results that lays certain limits on the theoretical possibilities for C.

The variation of B with Σ'_2, but not its variation with Σ'_3, can be found from the results of tests in simple shear, for in this case

$$e_{12} = B(\Sigma'_2, 0)\, \sigma_{12}.$$

If in fact the power law relation $e_{12} = (\sigma_{12}/A)^n$ holds, then

$$B(\Sigma'_2, 0) = (\Sigma'_2)^{\frac{1}{2}(n-1)}/A^n.$$

An interesting series of predictions concerning the behaviour of glaciers can be made by considering the effect of adding a small stress to a large stress already present. If equation (5) were true, then it could be said that the small addition does not affect Σ'_2 very much, particularly if the added stress components are not adding directly to stress components already present — if a compression is added to a body deforming by shear, for example. Thus if equation (5) were true, the material would react to this new stress exactly as though it were a Newtonian liquid of viscosity

$$\eta = 1/2B(\Sigma'_2).$$

If in addition the power law relation holds, then we can use equation (6) to write

$$\eta = A^n/2(\Sigma'_2)^{\frac{1}{2}(n-1)} \qquad (7)$$

This equation is similar to one derived by Nye ([10]) to explain the flow in the upper layers of the Junfraufirn; and there are several examples in glacier theory where this situation occurs, and we shall consider some of them briefly below, but first we ought to consider what effect the failure of equation (5) would have.

First we can note that it will be true to assume that neither Σ'_2 nor Σ'_3 will change greatly in magnitude, though it is of course possible that Σ'_3 might change from zero to some small quantity, as will occur in the case mentioned above where a small uniaxial compression is added to a larger shear stress. Thus we shall expect the effect of the small addition on the first term in equation (4) to be small. The second term will give an apparent Newtonian viscosity exactly as on the simpler theory, with exception that it may be added to an already present first term. This means that the added deformation will be apparently Newtonian provided the third term is small. This third term, however, if it is large, will give an added flow that is essentially non-Newtonian since in general it is proportional to the product of two stresses. We thus see that the general ideas derived on the assumptions contained in the simpler theory may not be strictly true if the contribution of the third term in equation (4) is large. With this warning, we can now consider the individual cases.

(a) *The closing of tunnels in rapidly deforming ice*. Nye ([10]) has shown that the rate of closing of tunnels dug into relatively smooth glaciers can be predicted using the flow law determined in the laboratory on the assumption that the closure is due to the weight of the overlying ice; in this theory he assumes that equation (5) is valid. However in those cases where the tunnel is below an ice fall ([10]) ([11]) the rate of closure is far larger. This effect can be understood if we realise that at such regions of a glaciers there is a large longitudinal stress acting in the ice, which, throughout its depth is undergoing a rapid compression ([11]). If we assume that this stress in the ice is large compared with the stresses due to the presence of the tunnel, and that the tunnel does not seriously modify the larger stress, then we can apply the ideas of a quasi-viscous closure of the tunnel and use the normal viscous closure law

$$s = p/2\eta,$$

where s is the closure rate and p the overburden pressure, the appropriate viscosity being given by equation (7). We thus expect the closure of such a tunnel to be pro-

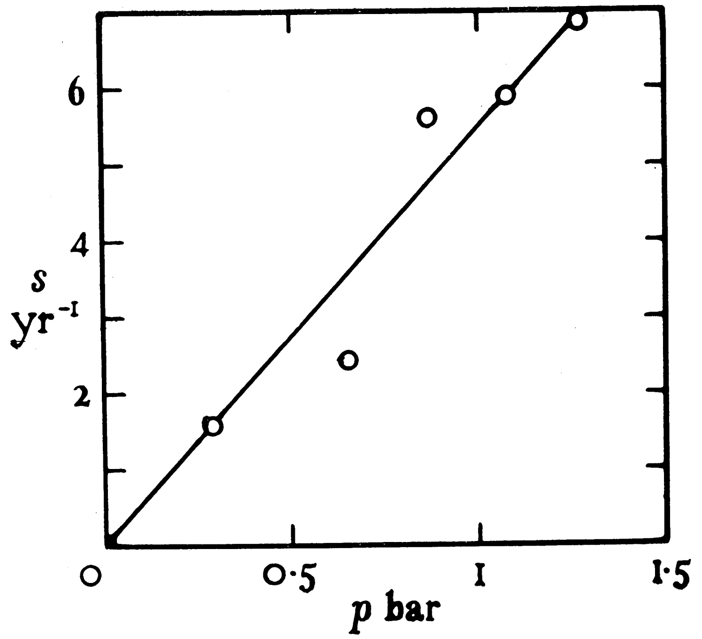

Fig. 3 — A graph of the rate of closure of the tunnel in Austerdalsbreen against the pressure due to the overlying ice.

portional to p, and for the Austerdalsbre tunnel ([11]). this is approximately true as can be seen in Fig. 3. Since detailed measurement of the deformation allowed the stress deviator to be calculated for the ice round this tunnel, we can use equation (7) to predict the slope of this line. Unfortunately, this does not give good agreement; the viscosity calculated from the stress deviator is 2.3×10^{14} poise, whereas the value needed to account for the tunnel closure using the viscous theory is 3×10^{12} poise. This disagreement could be due to a failure of the assumption that everywhere the stresses caused by the tunnel are small compared with those due to the flow of the glacier, for as soon as this assumption breaks down, the result is a rapid increase in the rate of flow, or a rapid decrease in the apparent viscosity.

(b) *Disappearance of surface irregularities in rapidly deforming ice.* In exactly the same way as we considered in the case of the tunnels, any other irregularity in the glacier which creates stresses round itself will cause the ice to flow more rapidly if large stresses are already present. Thus a crevasse or a surface wave will tend to disappear due to the flow of ice under its own weight, and these effects will be much larger, and will be calculable on a simple viscous theory, provided that, due to flow, the ice in the neighbourhood has large stresses in it already. Again the place where we would expect these effects to be most marked is below ice falls, for it is here that a large longitudinal compressive stress develops in the ice. This effect may act in addition to the other mechanisms discussed by Nye ([12]) which can cause the disappearance of wave ogives.

(c) *Flow of the ice past obstructions in the bed.* According to Weertman's theory of the slip of a glacier on its bed ([13]), a glacier overcomes the larger obstacles by flowing past them. This flow will be due to the stress concentrations caused by the obstacles, which will be superposed on any stresses already present in the ice. Unlike the case of irregularities on the surface, any flowing glacier will have quite large stresses due to flow in the neighbourhood, since the shear stress on the bed of most flowing, temperate glaciers is about the same ([14]), This effect will be of similar importance in most glaciers, but very fast flowing glaciers, or glaciers undergoing rapid longitudinal compression, may have somewhat larger flow stresses and hence be able to flow past obstacles more easily. This may help to explain the greater erosive power of glacier below ice falls. Another possible result of this theory, is that the value of n in equation (5) of Weertman's paper ought perhaps to be less than the value in the simple flow law, while the value of B should be greater.

(d) *Insertion of a ball into the ice.* Haefeli ([15]) has attempted to measure the «viscosity» of glacier ice by pressing a ball into it and measuring the rate at which it sinks. It is obvious from what we have said that this can only be expected to give an answer which is independent of the load, provided that all the loads used are sufficiently small for the stress they impart to the ice to be small compared with the stresses caused by glacier flow. In Haefeli's case this condition was not satisfied, for the ice was in the Z'Mutt Glacier, where the flow is quite smooth, and the stresses under the ball were up to 10 Kg./cm.^{2}; it is therefore not very surprising that his results do not give a straight line through the origin when sinking velocity is plotted against load, but in principle this method could be used with smaller loads to determine the value of the function B in rapidly flowing ice.

(e) *Flow of the upper layers of a glacier in compressing or extending flow.* In a glacier which has large longitudinal stresses either tensile or compressive, the shear stress caused by the weight of overlying ice is, at least in the upper layers, a small addition to the large longitudinal stress. (Nye ([10]) has considered this case, and he showed that the flow in the upper layers of the Jungfraufirn could be accounted for quite accurately by regarding it as due to a superposition of a longitudinal strain caused by the tensile stress and a shear strain computed using an effective viscosity determined by the longitudinal stress. Full details of this calculation, which is generally applicable to glaciers undergoing extending or compressive flow, are given in Nye's paper, and it is mentioned here solely to show that the idea introduced by Nye of an effective viscosity for the effects of small added stresses is an example of the law considered here, and that it has the other applications mentioned in sections(a) to (d) above.

Acknowledgements

I should like to than Dr. F. Ursell for introducing me to the general relations between second rank tensors contained in equations (2) to (4), and also Dr. S. Steinemann for allowing me to use unpublished experimental data.

APPENDIX

(A) Derivation of Equation (2).

In an isotropic meterial, or in a material whose anisotropy is entirely due to the action of the stress, and which must therefore have the same symmetry as the stress, the flow tensor must also have the same symmetry as the stress. Thus the individual components of the strain rate tensor must be related to the stress tensor by terms which can contain an arbitrary function of the invariants of the stress tensor, but can only otherwise contain the stress components in the form

$$e_{ij} = A(\Sigma_1, \Sigma_2, \Sigma_3)\,\delta_{ij} + B(\Sigma_1, \Sigma_2, \Sigma_3)\,\sigma_{ij} + C(\Sigma_1, \Sigma_2, \Sigma_3)\,\sigma_{ik}\,\sigma_{kj} + \\ + D(\Sigma_1, \Sigma_2, \Sigma_3)\,\sigma_{ik}\,\sigma_{kl}\,\sigma_{lj} + E(\Sigma_1, \Sigma_2, \Sigma_3)\,\sigma_{ik}\,\sigma_{kl}\,\sigma_{lm}\,\sigma_{mj} + \dots$$

The fourth and further terms can be shown to be expressible in terms like the first three as a result of the Cayley-Hamilton Theorem in matrix algebra. In our notation this theorem can be written

$$\sigma_{ik}\,\sigma_{kl}\,\sigma_{lj} - \Sigma_1\,\sigma_{ik}\,\sigma_{kj} - \Sigma_2\,\sigma_{ij} - \Sigma_3\,\delta_{ij} = 0,$$

and so can be used to rewrite these terms in the form of the earlier ones. This means that the most general form for the relation is that written as equation (2).

(B) Derivation of the flow law resulting from Nye's assumptions

In this section we shall consider the effect on equation (4) of making either or both of the following assumptions: (i) that the components of strain rate are proportional to the components of the stress deviator; (ii) that the second invariant of the strain rate tensor is a function of the second invariant of the stress deviator only.

Assumption (i) implies that the principal strain rates are proportional to the principal stress deviators.
This can be written $e_1/\sigma'_1 = e_2/\sigma'_2 = e_3/\sigma'_3$, and substituting for e_1, e_2 and e_3 from equation (4)

$$-\frac{2}{3}\Sigma_2'\,C(\Sigma'_2, \Sigma'_3)/\sigma'_1 + B(\Sigma'_2, \Sigma'_3) + C(\Sigma'_2, \Sigma'_3)\,\sigma'_1$$

$$= -\frac{2}{3}\Sigma'_2\,C(\Sigma'_2, \Sigma'_3)/\sigma'_2 + B(\Sigma'_2, \Sigma'_3) + C(\Sigma'_2, \Sigma'_3)\,\sigma'_2$$

$$= -\frac{2}{3}\Sigma'_2\,C(\Sigma'_2, \Sigma'_3)/\sigma'_3 + \mathrm{B}(\Sigma'_2, \Sigma'_3) + \mathrm{C}(\Sigma'_2, \Sigma'_3)\,\sigma'_3,$$

and $B(\Sigma'_2, \Sigma'_3)$ can be subtracted from each of these expressions. After this has been done, $C(\Sigma'_2, \Sigma'_3)$ occurs in all terms, and so can be divided out provided it is not zero. The resulting expression is

$$-\frac{2}{3}\Sigma'_2/\sigma'_1 + \sigma'_1 = -\frac{2}{3}\Sigma'_2/\sigma'_2 + \sigma'_2 = -\frac{2}{3}\,\Sigma'_2/\sigma'_3 + \sigma'_3.$$

But this equation is clearly not generally true, for example it is not true if $\sigma'_1 = 1$, $\sigma'_2 = -1$ and $\sigma'_3 = 0$. Thus the assumption that $\mathrm{C}(\Sigma'_2, \Sigma'_3)$ is not zero must be false, and so equation (4) reduces to

$$e_{ij} = B(\Sigma'_2, \Sigma'_3)\,\sigma_{ij}. \tag{8}$$

If in addition assumption (ii) holds, then

$$E_2 = \frac{1}{2}\,e_{ij}\,e_{ij} = f(\Sigma'_2), \tag{9}$$

and if we substitute for e_{ij} from (8), we obtain

$$E_2 = B^2(\Sigma'_2, \Sigma'_3)\,\Sigma'_2,$$

and the right hand side of this must, according to (9), be a function of Σ'_2 only, so that B must be a function of Σ'_2 only, which reduces equation (8) to equation (5).

If finally we consider the effect of assumption (ii) without assumption (i), we can substitute for e_{ij} in equation (9) from equation (4), and then, after certain algebraic rearrangements, using the Caley-Hamilton Theorem and the fact $\Sigma'_1 = 0$, we obtain

$$E_2 = B^2(\Sigma'_2, \Sigma'_3)\,\Sigma'_2 + 3\,B(\Sigma'_2, \Sigma'_3)\,C(\Sigma'_2, \Sigma'_3)\,\Sigma'_3 + \frac{1}{3}\,C^2(\Sigma'_2, \Sigma'_3)(\Sigma'_2)^2,$$

and if this is a function of Σ'_2 only, then either B and C must have a very special form or else one of B and C must be zero and the other must be a function of Σ'_2 only. However if B is zero, and C is a function of Σ_2 only, then the rate of working is

$$e_{ij}\,\sigma_{ij} = 3\,C(\Sigma'_2)\,\Sigma'_3,$$

and if the sign of all the stress components is changed, Σ'_2 does not change. and so C does not change, but Σ'_3 changes sign. However the rate of working must always be positive and so this particular possible result of the assumption is shown to be false. This suggests that assumption (ii) by itself makes (5) very likely, and if (5) breaks down for any initially isotropic material, only two possibilites seem likely. These are first that assumption (i) remains true so that C is zero and B a function of both Σ'_2 and Σ'_3, and second that equation (4) has to be used in full. Experiments to investigate the presence of expansion or contraction in shear tests of the kind considered in the section on applications to the theory of glacier flow, would distinguish between these two possibilities.

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