THE PATTERN OF FLOW IN THE VICINITY OF A RECHARGING AND DISCHARGING PAIR OF WELLS IN AN AQUIFER HAVING AREAL PARALLEL FLOW *

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SUMMARY

To determine the pattern of flow in the vicinity of a pair of wells of which one is being recharged and one is discharging is important in solving certain ground-water problems, such as those involved in the application of tracers and the disposal of liquid contaminants. The mathematical relationships of the flow characteristics are of significance in the design of a well disposal system, particularly where waste liquids are to be disposed of near pumped water-supply wells.

Through the application of functions of a complex variable the general equations are derived for the steady-state flow pattern in the vicinity of a recharging-discharging pair of wells in an aquifer having areal parallel flow. The equations can be used to determine the quantity of interflow between the wells for any recharge and discharge rates and for different angles between the line joining the wells and the line of natural areal flow.

Flow patterns are given for examples in which the recharge-discharge rates are equal and in which the wells are aligned at various angles to the direction of areal flow. The flow patterns for these examples indicate that the interception of water from the recharge well by the discharge well is not at a minimum where the recharge well is directly downgradient from the discharge well; instead, the interflow may be least where the two wells (recharge well downstream) are aligned at a substantial angle to the direction of areal flow. The mathematical solutions of these examples were verified by graphical construction of flow nets.

RÉSUMÉ

Le système d'écoulement plan à proximité d'un puits d'alimentation (source) et d'un puits est très important pour résoudre certains problèmes d'eaux souterraines par exemple ceux concernant les applications des traceurs et la destruction des déchets. Les expressions mathématiques qui décrivent les caractéristiques de l'écoulement sont importantes dans la construction d'un système de destruction des déchets, en particulier quand les déchets liquides sont éliminés dans les environs de puits de distribution. Avec l'aide des fonctions de variable complexe on a écrit les équations générales du système de l'écoulement plan permanent autour d'une paire de puits situés dans une nappe aquifère avec écoulement parallèle. Ces équations peuvent être utilisées pour déterminer la circulation entre les deux puits quelle que soit la proportion du débit d'alimentation et pour des angles différents entre la ligne joignant les puits et la direction de l'écoulement parallèle.

Des exemples de reseaux d'équipotentielles et de lignes de courant sont donnés; dans ces exemples les débits du puits et de la source sont les mêmes et les deux puits sont alignés avec des angles variés sur la direction de l'écoulement parallèle. Les réseaux pour ces exemples indiquent que l'interception de l'eau de la source par le puits n'est pas au minimum quand la source est placée directement à l'aval du puits; au lieu de cela, la circulation entre les deux puits peut être moindre quand les deux puits sont alignés avec un angle substantiel sur la direction de l'écoulement parallèle. Les solutions mathématiques de ces exemples ont été vérifiées par construction géométrique des réseaux d'équipotentielles et de lignes de courant.

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1. Introduction

The relationships describing the patterns of flow in the vicinity of recharging and discharging wells are of significance in ground-water development and utilization because of the need to dispose of various liquid contaminants, such as some types of radioactive waste, and the return of ground water after it has been warmed from use in air conditioning.

The problem is primarily one of designing a recharge-discharge well system so that the liquid injected into a recharge well will not reach nearby pumped water-supply wells.

All aquifers have some natural areal flow through them; consequently the flow patterns around recharging and discharging wells are affected by the strength and direction of the areal flow.

The steady-state flow formed by a pair of recharging and discharging wells in an aquifer having natural parallel flow is identical to the flow system, source and sink superimposed on a field of parallel flow, of classical hydrodynamics (Milne-Thomson, 1938, p. 198, 202, 204) (*) Jacob (1950, p. 348-351) analyzed the problem for a recharge well situated directly downgradient from the discharge well and determined the relationship of the quantity of interflow between the wells and the ratio of the rate of recharge and discharge to the areal flow rate and the distance between the wells.

Dr. C.V. Theis of the U.S. Geological Survey (personal communication, 1959) recently analyzed the problem by use of graphical methods and found that there is less interflow between the wells if the wells are aligned at an angle to the direction of areal flow, with the recharge well downstream. At the suggestion of Dr. Theis the senior author determined mathematically the relationship of the various factors affecting the interflow between the wells. The expressions relating those factors and flow nets, constructed graphically, for various conditions of interflow are given in this paper.

2. Definition of the flow system

For the problem considered in this paper the discharge and recharge rates of the two wells have the same value (Q). The discharge well is one the x-axis at a point with coordinates $(-a, 0)$ and the recharge well at a point on the x-axis $(+a, 0)$ (fig. 1). If the flow system created by the pair of discharge and recharge wells is superimposed on a natural system of areal parallel flow of velocity (v_0) whose orientation is at an angle (α) with the x-axis, measured counterclockwise from that axis, the resulting flow system may be represented by the complex potential W and described by the relation:

$$W = Uz - \frac{Q}{2\pi} \, ln(z - a) + \frac{Q}{2\pi} \, ln(z + a) \qquad (1)$$

in which

$$U = v_0 e^{-i\alpha} = v_0(\cos\alpha - i \sin\alpha) \qquad (1a)$$

and

$$z = x + iy$$

(*) See references at end of paper.

Equation (1) may be written:

$$W = -v_0(\cos\alpha - i\sin\alpha)(x + iy) + \frac{Q}{2\pi} ln[(x + a)^2 + y^2]^{1/2} +$$

$$+ \frac{iQ}{2\pi} \tan^{-1} \frac{y}{x + a} - \frac{Q}{2\pi} ln[(x - a)^2 + y^2]^{1/2} - i\frac{Q}{2\pi} \tan^{-1} \frac{y}{x - a} \qquad (2)$$

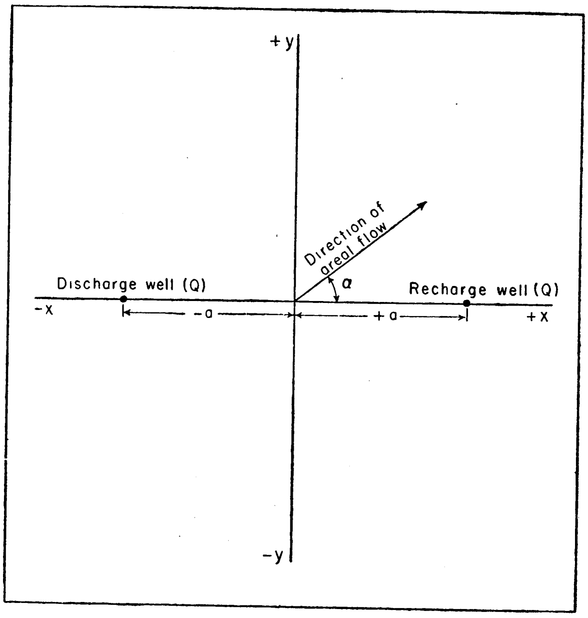


Fig. 1 — Diagram showing position of recharge and discharge wells with respect to the coordinate axes and the method of measuring α, the angle between the direction of areal flow and the wells.

As $W = \phi + i\psi$, in which ϕ is the velocity potential and ψ the stream function, equation (2) after simplifying, becomes:

$$\psi = -v_0(y\cos\alpha - x\sin\alpha) + \frac{Q}{2\pi} \tan^{-1} \frac{2ay}{a^2 - x^2 - y^2} \qquad (3)$$

and

$$\phi = -v_0(x\cos\alpha + y\sin\alpha) + \frac{Q}{4\pi} ln \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2} \qquad (4)$$

Equations (3) and (4) show that the stream function and the velocity potential are dependent upon the ratio of the recharge-discharge rate to the areal flow rate, the angle between the direction of areal flow and the line joining the wells, and the distance between the two wells.

The interflow between the wells is dependent on the pattern of flow between them and thus is in turn dependent on the values of the stream function and the velocity potential.

The conditions governing the interflow between the wells are characterized by the location of the stagnation points that is, the points at which the velocity is zero.

The stagnation points are thus the points where

$$\frac{dW}{dz} = 0$$

Hence, from (1) and (1a):

$$\frac{dW}{dz} = - v_0 e^{-i\alpha} + \frac{Q}{2\pi(z+a)} - \frac{Q}{2\pi(z-a)} = 0 \tag{5}$$

or, after solving for Z and simplifying:

$$z = x + i\,y = a \sqrt{1 - \frac{Q}{\pi a v_0} (\cos\alpha + i \sin\alpha)} \tag{6}$$

which is the general expression for the location of the stagnation points.

The values of x and y, the coordinates of the stagnation points, are given by the following expressions:

$$x = \pm \sqrt{\frac{1}{2}\left[\left(a^2 - \frac{aQ}{\pi v_0}\cos\alpha\right) + a\sqrt{a^2 + \left(\frac{Q}{\pi v_0}\right)^2 - 2\,\frac{aQ}{\pi v_0}\cos\alpha}\right]} \tag{7}$$

and

$$y = \pm \sqrt{\frac{1}{2}\left[\left(\frac{aQ}{\pi v_0}\cos\alpha - a^2\right) + a\sqrt{a^2 + \left(\frac{Q}{\pi v_0}\right)^2 - 2\,\frac{aQ}{\pi v_0}\cos\alpha}\right]} \tag{8}$$

Equations (7) and (8) usually yield two pairs of values for x and y; these values, however, cannot be combined in an arbitrary manner. The permissible pairs of x and y are those of opposite signs.

A graph showing the locations of the stagnation points for the different values of $\frac{Q}{\pi a v_0}$ and α, is shown in figure 2. The angle α, at which the values of y are at a maximum, also is shown. The maximum values of y, of course, indicate the locations at which the stagnation points are at the greatest distance from the x-axis, the line through the pair of wells.

The maximum values of y correspond also to the angle α_m which, for a given ratio $\frac{Q}{\pi a v_0}$, is that at which the interflow between the wells is at a minimum.

The maximum of y is obtained from equation (8) by determining the derivative $dy/d\alpha$ and equating it to zero. Performing this operation and simplifying produce the relation:

$$\cos\alpha_m = \frac{Q}{2\pi a v_0} \tag{9}$$

which is the general expression for the angle at which the interflow between the

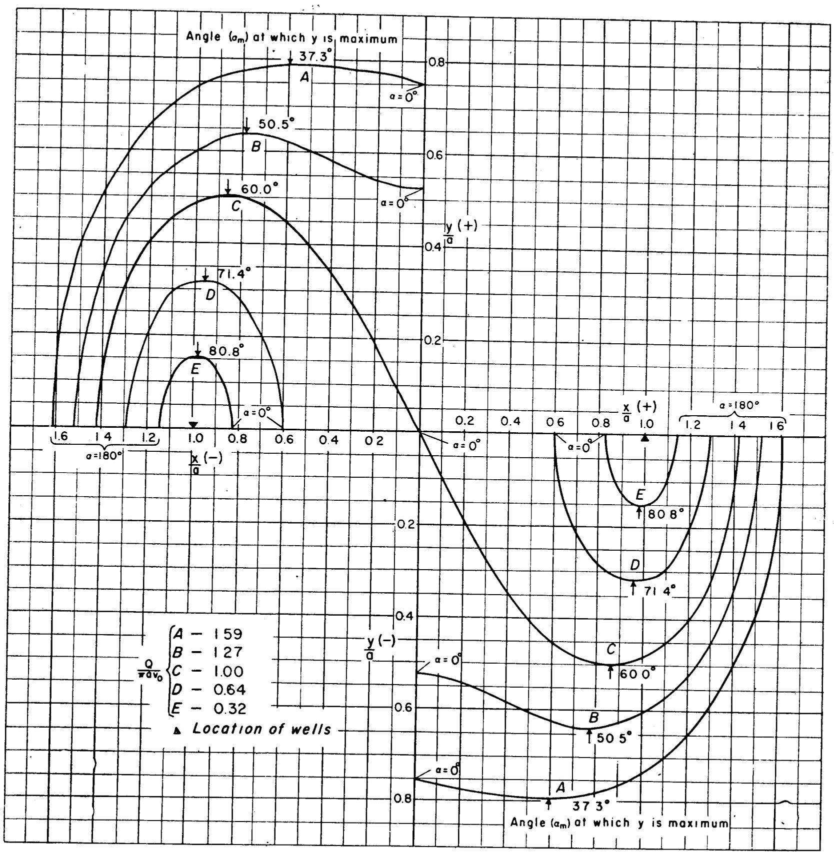


Fig. 2 — Curves showing location of stagnation points for different values of $\dfrac{Q}{\pi a v_0}$ and the angle (α) between the direction of areal flow and the line formed by the recharging and discharging pair of wells.

recharge and discharge well is at a minimum. Thus from (9) the cosine of the angle α_m varies directly with the ratio $\dfrac{Q}{v_0}$ and inversely with the half-distance (a) between the two wells.

A curve showing the relation of α_m to different values of $\dfrac{Q}{\pi a v_0}$ is shown in fig. 3

The quantity of interflow (I) between the wells, as a percentage of Q, is given by multiplying by 2 the difference between the value of the stream function at the origin of coordinates ($\psi_{0,0}$) and the value of the stream function at one of the stagnation points (ψ_s); that is:

$$I = 2(\psi_{0,0} - \psi_s) \tag{10}$$

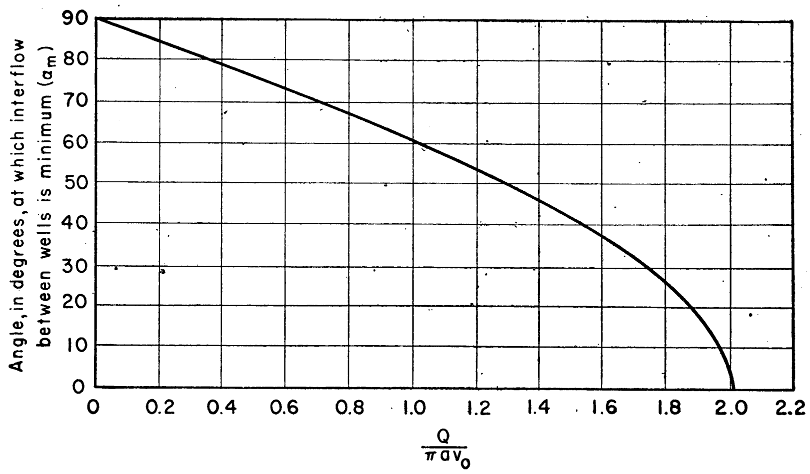


Fig. 3 — Relation between angle (α_m), at which interflow between wells is at a minimum, and $\frac{Q}{\pi a v_0}$.

Values of $\psi_{0,0}$ and ψ_s for different values of $\frac{Q}{\pi a v_0}$ are obtained from equation (3) by substituting therein the coordinate values of the corresponding stagnation points.

The interflow (I) so derived may be plotted against different angles (α) for different values of $\frac{Q}{\pi a v_0}$. A set of such curves is given in figure 4. The curves show that the amount of interflow decreases with $\frac{Q}{\pi a v_0}$ and that the angle (α_m) at which the interflow is at a minimum also varies with different values of $\frac{Q}{\pi a v_0}$ (*). The curve for $\frac{Q}{\pi a v_0} = 1.59$ shows that the interflow is about 11 percent where α is 0_0 —that is, when the recharge well is directly downstream from the discharge well, and decreases to a minimum of 10 percent at an angle (α_m) of 37.3°. The interflow is 100 percent, as it is for all values of $\frac{Q}{\pi a v_0}$, at an angle (α) of 180° — that is, where the recharge well is directly upstream from the discharge well.

The curve for $\frac{Q}{\pi a v_0} = 1.27$ shows that, except for one point at which the interflow is zero, there is positive interflow between the wells. The curves for smaller values of $\frac{Q}{\pi a v_0}$ show

(*) As considered in this paper the interflow may be less than zero; that is, the pattern of flow is such that flow lines representing the natural areal flow pass between the two wells. Thus the values of negative interflow are quantitative representations of the number of such flow lines separating the flow between the wells.

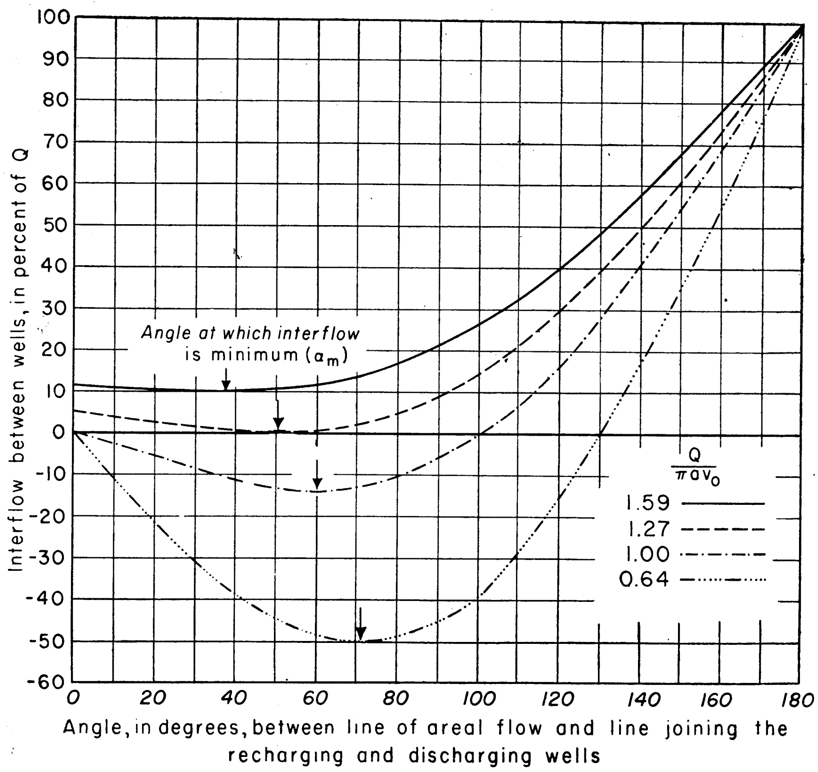


Fig. 4 — Graph showing the relationships of the interflow between the wells and the angle (*I*) formed by the areal-flow direction and the recharging and discharging pair of wells for different values of $\frac{Q}{\pi a v_0}$.

that, for segments of the curves corresponding to certain ranges of angle α, there is negative interflow. Thus the value of $\frac{Q}{\pi a v_0} = 1.27$ is a limiting condition; that is, there is positive interflow for values of $\frac{Q}{\pi a v_0}$ larger than 1.27 at all angles α, and there is negative interflow for smaller values of $\frac{Q}{\pi a v_0}$ within certain ranges of angle α.

3. Flow nets

Flow nets showing different conditions of interflow were constructed graphically by superimposing the flow nets for a pair of recharging and discharging wells (fig. 5) on a grid representing areal parallel flow. The resulting flow nets, for different angles (α) and for $\frac{Q}{\pi a v_0} = 1.27$ and 1.0, are shown in figures 6-9. The following table gives the pertinent values for the flow nets.

$$\frac{Q}{\pi a v_0} = 1.27$$

α (degrees)	$\frac{x}{a}$	$\frac{y}{a}$	ϕ (% Q)	ψ (% Q)	I (% Q)
0	0.000	$\pm$ 0.523	0.000	$\pm$ 0.002	+ 4.4
50.5	$\mp$ .722	$\pm$ .644	$\mp$ .164	— .500	0.0
90	$\mp$ 1.144	$\pm$ .556	$\mp$ .355	$\mp$ .455	+ 9.0
180	$\mp$ 1.521	.000	$\mp$ .632	.000	+ 100.0

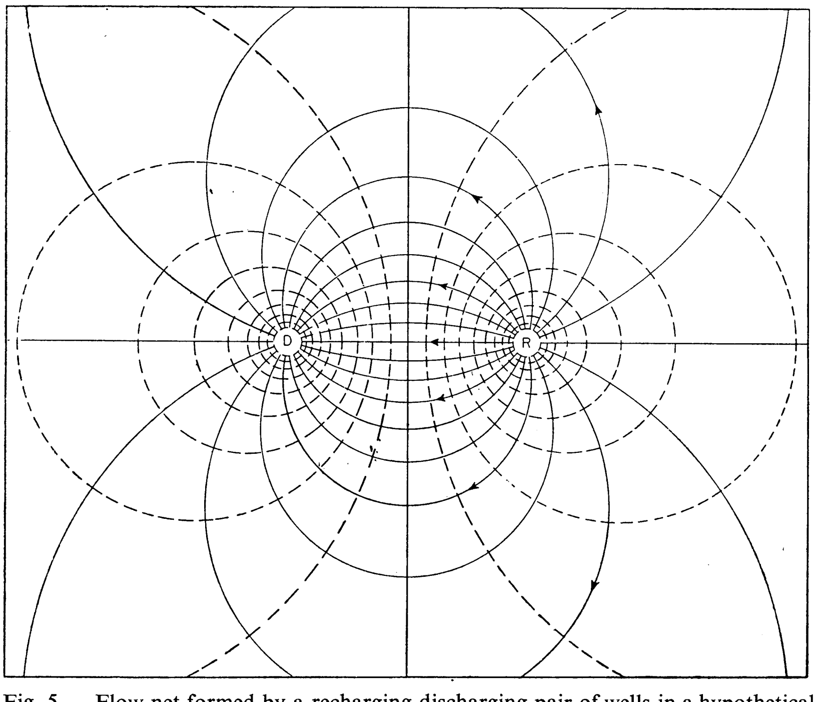


Fig. 5 — Flow net formed by a recharging-discharging pair of wells in a hypothetical aquifer having no areal flow.

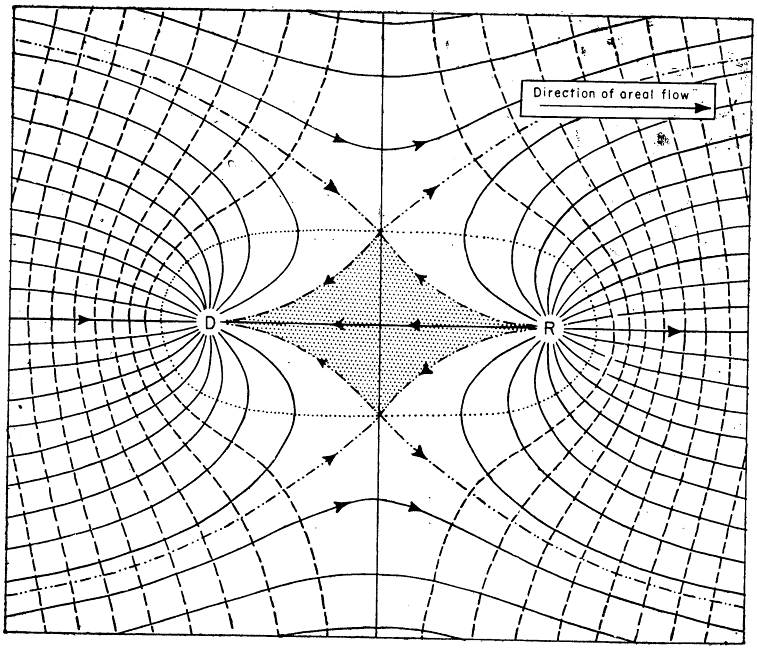
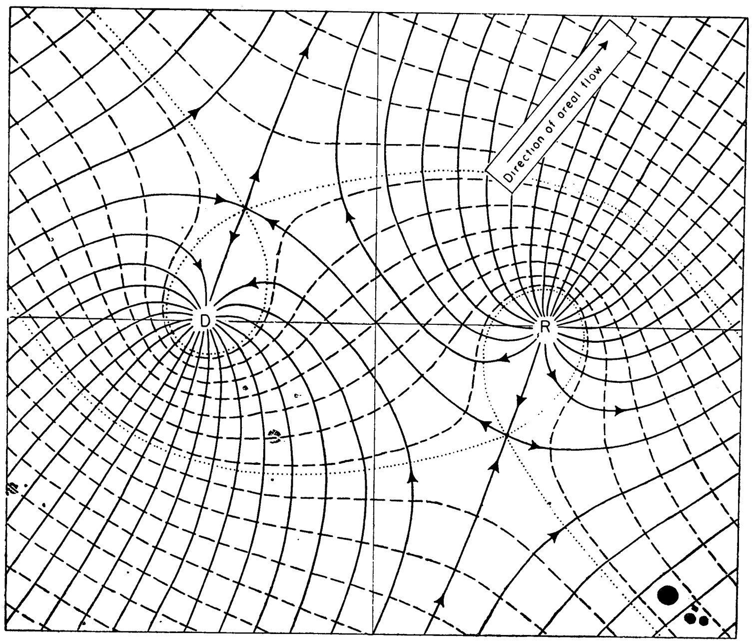


Fig. 6 — Flow net for recharging-discharging pair of wells with $\frac{Q}{\pi a v_0} = 1.27$, A-angle $\alpha = 0°$, B-angle $\alpha = 50.5°$.



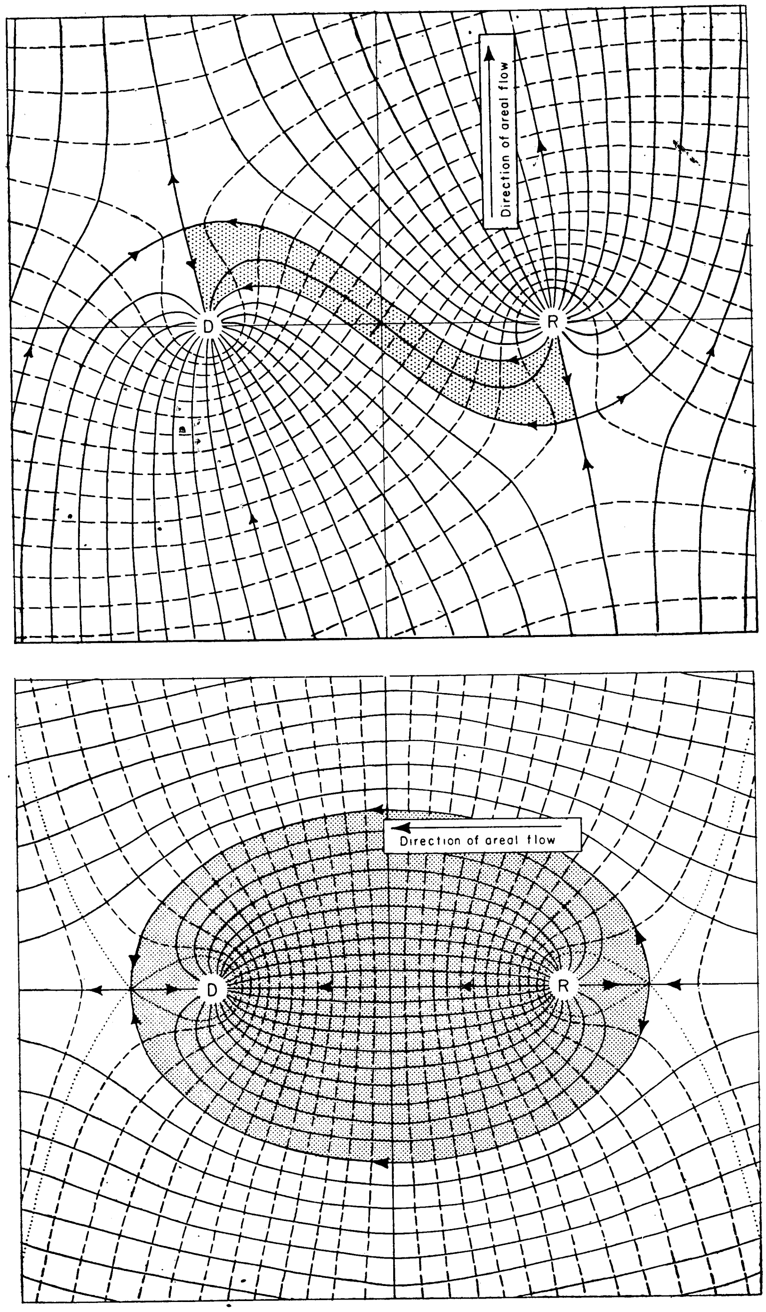
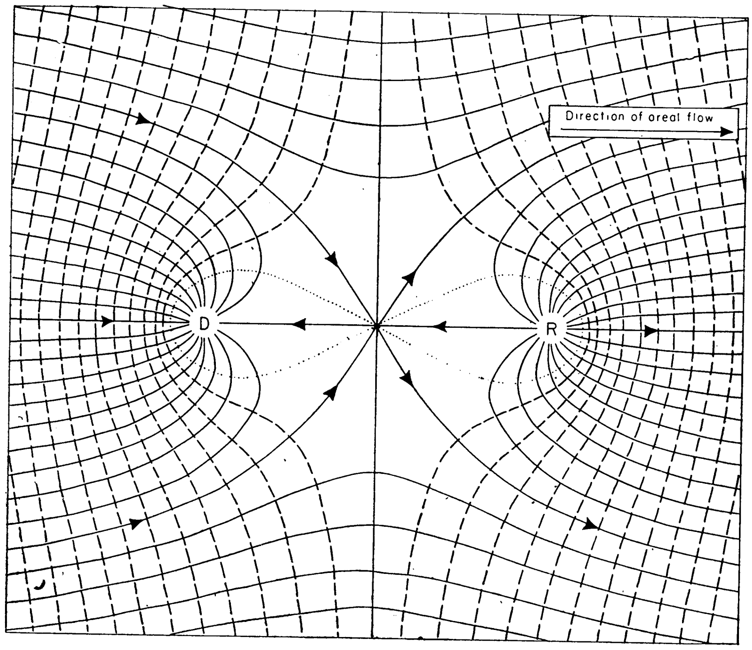


Fig. 7 — Flow net for recharging-discharging pair of wells with $\frac{Q}{\pi a v_0} = 1.27$, A-angle $\alpha = 90°$, B-angle $\alpha = 180°$.



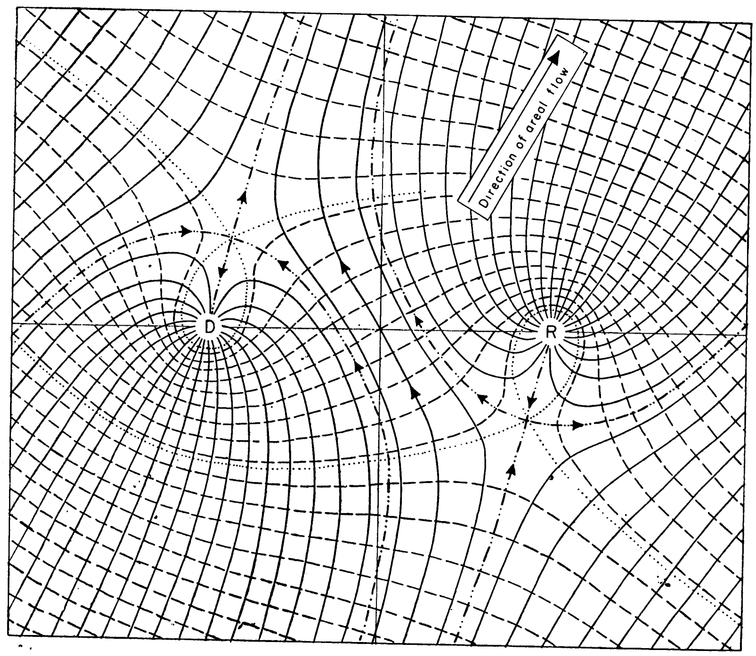


Fig. 8 — Flow net for recharging-discharging pair of wells with $\frac{Q}{\pi a v_0} = 1.0$, A-angle $\alpha = 0°$, B-angle $\alpha = 60°$.

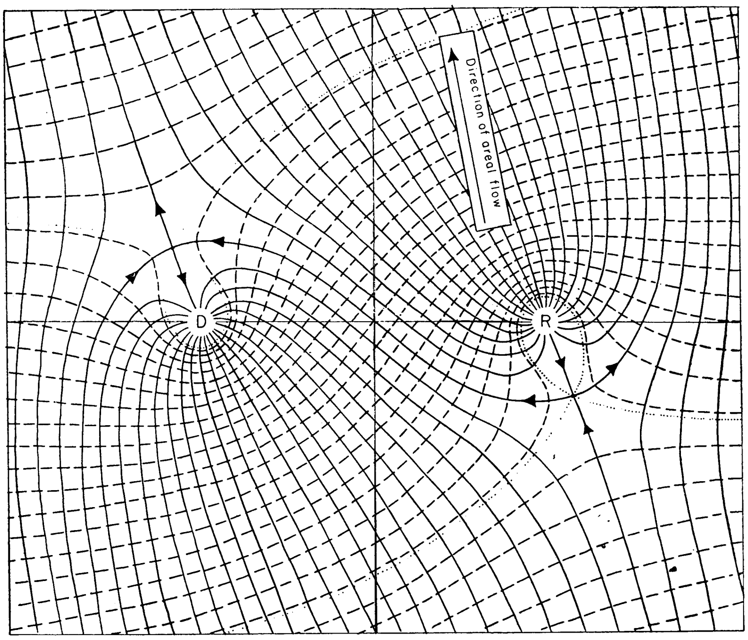


Fig. 9 — Flow net for recharging-discharging pair of wells with $\frac{Q}{\pi a v_0} = 1.0$ and angle $\alpha = 101°$.

$$\frac{Q}{\pi a v_0} = 1.0$$

α (degrees)	$\frac{x}{a}$	$\frac{y}{a}$	ϕ (% Q)	ψ (% Q)	I (% Q)
0	0.000	0.000	0.000	0.000	0.0
60	± .867	± .500	± .210	— .070	— 14.0
101	± 1.169	± .420	± .461	— .500	.0

4. Conclusions

For problems concerned with preventing interflow between recharging and discharging wells, such as those involving disposal of liquid contaminants and the injection of warm water returned after use in air conditioning, the recharge well should be downgradient from the discharge well. The best downgradient direction, however, is not parallel to the direction of areal flow as might be supposed, but is at some angle to that direction. The ratio of the well recharge-discharge ratel (assuming them equal) to the areal flow rate and the distance between the wells also are of significance in preventing interflow. According to the recharge-discharge rate, the areal flow rate, and the distance between the wells, interflow between the wells can occur at all angles regardless of whether the recharge well is downgradient from the discharge well. The interflow can be decreased or prevented entirely by decreasing the recharge-discharge rate, by increasing the distance between the wells, by aligning the wells at certain angles to the direction of flow, or by a combination of these methods.

REFERENCES

JACOB, C.E., (1950), Flow of ground water, *in* Rouse, Hunter (ed.), Engineering hydraulics: New-York, John Wiley and Sons; London, Champan and Hall Ltd.

MILNE-THOMSON, L.M., (1938), Theoretical hydrodynamics: London, MacMillan Co.