

## **A mathematical model of water movement through porous media**

**YURIY DENISOV, ALEXEY SERGEEV**

*SANIGMI of Glavgidromet, 72 K. Maksumov Str., 700052 Tashkent, Uzbekistan*

**GERMAN BEZBORODOV**

*Uzbek National Cotton Growing Research Institute, PO Box Akkavak, Kibray district, Tashkent region 702133, Uzbekistan*

**YURIY BEZBORODOV**

*Russian Hydrotechnic and Land Reclamation Research Institute, 44 Bolshaya Academicheskaya, Moscow 127550, Russia*

**Abstract** It is assumed that water flow in porous media is proportional to the gradient of some potential, multiplied by either the saturated hydraulic conductivity coefficient or the unsaturated hydraulic conductivity coefficient. This is true if the porosity of the medium is not very high and its specific surface is not very small. However, for some porous substances such as gravels and fractured rocks, and also for soils penetrated by the burrows of earthworms and other soil animals, the law of the linear relationship between water flow and potential is disturbed. For the generalization of the water flow in media with a dual porosity, it is expedient to deduce the general expression in a vector form, as in the general case, the spatial problem is considered. Theoretical investigations of the water flow in porous media on the basis of the mechanics of the multiphase and multicomponent media, have made it possible to get the mathematical model of the water flow in the considered media. The model is presented by the equations of movement of phases and mass balance in vector form. The expression for the saturated hydraulic conductivity coefficient has been obtained with the characteristics of vital activity of the vital phase in porous media being taking into account. The numerical realization of the model was made for serozem soils in central Asia. Comparison between the calculated values and the measured ones have shown the acceptable accuracy of the model and its applicability for practical purposes.

### **INTRODUCTION**

Many investigators, including hydrogeologists and soil scientists, face the problem of water movement through porous media. It is usually considered that the water flow in such media is proportional to a gradient of some potential which is multiplied by either the saturated hydraulic conductivity coefficient, or the unsaturated hydraulic conductivity, according to conditions. That is how matters stand if the porosity of the medium is not very high, and the specific surface is not very small. However, for some media such as pebbles, gravel, fractured rocks and soils pierced by burrows of earthworms and other animals, the law of the linear relationship between water flow and the potential breaks down. In such a situation it is desirable to derive the

expression for the water flow in a porous medium in vectorial form (as in the general case the spatial problem is studied) applicable both to light and heavy media; i.e. for media with weak permeability this expression would be practically linear concerning the gradient of potential, and for a medium with strong permeability it would be nonlinear. The derivation of the necessary expression which follows is based on the mechanics of multiphase and multicomponent media.

## THEORETICAL BASIS

### The main characteristics of a multiphase porous medium

Let us choose a small finite volume  $\Delta V$  in the porous medium. Let us introduce phase index  $i = \overline{1,3}$ , which is equal to 1 for the matrix of the medium, 2 for water, and 3 for humid air. Let  $\Delta V_i$  be the volume of  $i$ th phase in the chosen volume  $\Delta V$ , then:  $\Delta V = \sum_{i=1}^3 \Delta V_i$ . The relative phase volume is calculated as:

$$\alpha_i = \frac{\Delta V_i}{\Delta V} \quad (1)$$

In this case we obtain  $\sum_{i=1}^3 \alpha_i = 1$ ,  $0 \leq \alpha_i \leq 1$ .

Fundamental characteristics of the multiphase medium are the specific surface of the phases,  $\beta_i$  and contact between phases,  $\beta_{ij}$ , through which dynamic and thermodynamic interactions take place. Let  $\Delta S_{ij}$  be the contact surface of the  $i$ th phase with the  $j$ th in the volume  $\Delta V$ . Then:

$$\beta_{ij} = \frac{\Delta S_{ij}}{\Delta V} \quad (2)$$

The total specific surface of the  $i$ th phase is equal to  $\beta_i = \sum_{j=1}^3 \beta_{ij}$ . An important characteristic of the multiphase medium is the effective thickness of the phase,  $\delta_i$ , that is equal to the ratio of the phase volume  $\alpha_i \Delta V$  to its total surface  $\beta_i \Delta V$ , i.e.:

$$\delta_i = k \frac{\alpha_i \Delta V}{\beta_i \Delta V} = k \frac{\alpha_i}{\beta_i} \quad (3)$$

where  $k$  is a dimensionless parameter which varies basically from 2 to 3 (Denisov, 1968, 1978). Next, assume that the matrix is rigid (swelling of the soil matrix due to water uptake is not taken into consideration). Defining  $\beta_c = \beta_1$  i.e. the specific surface of the porous medium, according to equation (3) the relationship for the characteristic pore size,  $\delta_n$ , is:

$$\delta_n = k \frac{1 - \alpha_1}{\beta_c} \quad (4)$$

### The equation of water movement through a porous medium

Let us consider the two cases saturation states of porous medium: (a) the pores are completely filled by water and (b) the pores are not completely filled by water (e.g. in the unsaturated zone)

(a) **Pores are completely filled by water** ( $\alpha_3 = 0$ ). Consider the forces affecting the volume of water  $(1 - \alpha_1)\Delta V$  in the porous medium volume:

*Gravity*

$$\rho_2(1 - \alpha_1)\vec{g} \Delta V = -\rho_2 g(1 - \alpha_1) \text{grad } z \Delta V \quad (5)$$

where  $\rho_2$  is the density of water and  $z$  is the upward vertical coordinate.

*Pressure*

$$\left\{ -\text{grad}[(1 - \alpha_1)P_2] + P_2 \text{grad}(1 - \alpha_1) \right\} \Delta V = -(1 - \alpha_1) \text{grad } P_2 \Delta V \quad (6)$$

The second term on the left side of equation (6) represents the reaction of pore side walls to water pressure.

*Viscous friction force* This force equals the product of coefficient of the dynamic water viscosity and the derivative of water speed at the pore surface (speed is directed along the normal to this surface) and the area of this surface. This force is defined by the following expression:

$$-k_{12} \rho_2 \nu_2 \beta_c \frac{\vec{u}_2}{\delta_n} \Delta V = -\frac{k_{12}}{k} \rho_2 \nu_2 \frac{\beta_c^2}{(1 - \alpha_1)} \vec{u}_2 \Delta V \quad (7)$$

where  $\vec{u}_2$  is a vector of water speed,  $\nu_2$  is a coefficient of kinematic water viscosity, and  $k_{12}$  is a dimensionless coefficient of proportionality which is approximately equal to 3.

*Eddy friction force* The absolute value of this force is proportional to the density of water, the pore surface area and the water speed squared. The force is directed against the flow and equals:

$$-k_{22} \rho_2 \beta_c |\vec{u}_2| \vec{u}_2 \Delta V \quad (8)$$

*Inertia force* The inertia force is the product of the water mass and its acceleration and is negative

$$-(1 - \alpha_1) \rho_2 \Delta V \frac{d\vec{u}_2}{dt} = -(1 - \alpha_1) \rho_2 \left[ \frac{\partial \vec{u}_2}{\partial t} + (\vec{u}_2 \nabla) \vec{u}_2 \right] \Delta V \quad (9)$$

Adding all the forces together and dividing this sum by  $(1 - \alpha_1)\rho_2\Delta V$  and making some simple modifications, we obtain the equation of water movement in a porous medium when the pores are completely filled by water.

$$\frac{\partial \bar{u}_2}{\partial t} + (\bar{u}_2 \nabla) \bar{u}_2 = -g \operatorname{grad}(z + P_2^*) - \frac{k k_{12}}{\delta_n^2} \left( v_2 + \frac{k_{22}}{k_{12}} \delta_n |\bar{u}_2| \right) \bar{u}_2 \quad (10)$$

Here  $P_2^* = P_2/\gamma_2$ , where  $\gamma_2 = \rho_2 g$ .

After the analysis of equation (10) it was concluded that in this equation the inertia terms (left part) are significantly smaller than the other ones and, consequently, they can be ignored. Then:

$$\frac{k k_{12}}{\delta_n^2} \left( v_2 + \frac{k_{22}}{k_{12}} \delta_n |\bar{u}_2| \right) \bar{u}_2 = -g \operatorname{grad}(z + P_2^*) \quad (11)$$

It follows from equation (11) that the speed vector  $\bar{u}_2$  is co-linear to  $-g \operatorname{grad}(z + P_2^*)$ . So, according to vectorial calculation, we can write

$$\bar{u}_2 = -\lambda g \operatorname{grad}(z + P_2^*) \quad (12)$$

Here  $\lambda$  is a coefficient of proportionality which is larger than, or equal to, zero, and it is necessary to determine it.

Let us substitute equation (12) for equation (11). We can find the expression for coefficient  $\lambda$ , and in this case the equation (12) will be written as:

$$\bar{u}_2 = \frac{k_{12}}{2k_{22}} \cdot \frac{v_2}{\delta_n} \left[ \sqrt{1 + 4 \frac{k_{22}}{k_{12}^2} \cdot \frac{\delta_n^3}{k v_2^2} g |\operatorname{grad}(z + P_2^*)|} - 1 \right] \frac{\operatorname{grad}(z + P_2^*)}{|\operatorname{grad}(z + P_2^*)|} \quad (13)$$

Water flow through the unit of porous medium surface per unit of time when the pores are completely filled by water equals:

$$\bar{q}_2 = (1 - \alpha_1) \bar{u}_2 \quad (14)$$

or taking into consideration equation (13) we can obtain

$$q_2 = \frac{k_{12}}{2k_{22}} \cdot \frac{v_2 \beta_n}{k} \left[ \sqrt{1 + 4 \frac{k_{22}}{k_{12}^2} \cdot \frac{\delta_n^3}{k^2 v_2^2} g |\operatorname{grad}(z + P_2^*)|} - 1 \right] \frac{\operatorname{grad}(z + P_2^*)}{|\operatorname{grad}(z + P_2^*)|} \quad (15)$$

For negligible values of the second term of the expression under the root in equation (15) comparing with unit (linear flow), equation (15) is written as

$$\bar{q}_2 = -\frac{k(1 - \alpha_1)^3}{k_{12} \beta_c^2} \cdot \frac{\gamma_2}{\mu_2} \cdot \operatorname{grad}(z + P_2^*) = -K_\Phi \operatorname{grad}(z + P_2^*) \quad (16)$$

Equation (16) for saturated flow coincides with Darcy's equation (equation (1)) but the saturated hydraulic conductivity coefficient is defined in terms of the medium:

$$K_\Phi = \frac{k \gamma_2 (1 - \alpha_1)^3}{k_{12} \mu_2 \beta_c^2} = \frac{\gamma_2 \beta_c \delta_n^3}{k_{12} k^2 \mu_2} \quad (17)$$

It follows from equation (17) that the saturated hydraulic conductivity coefficient is proportional to the cube of porosity  $(1 - \alpha_1)^3$  and inversely proportional to the square of the specific surface and liquid viscosity  $\mu_2 \beta_c^2$ . Otherwise, this value can be considered proportional to the specific surface of the porous medium  $\beta_c$  and to the

cube of the pore radius  $\delta_n^3$  and inversely proportional to the liquid viscosity  $\mu_2$  (Nerpin & Chudnovskiy, 1967).

Let us define the first factor of equation (15) as  $u_g = \frac{k_{12} v_2 \beta_n}{2k_{22}k}$ , then the equation of water movement has the final form:

$$\bar{q}_2 = -u_g \left[ \sqrt{1 + 2 \frac{k_{\phi}}{u_g} |\text{grad}(z + P_2^*)|} - 1 \right] \frac{\text{grad}(z + P_2^*)}{|\text{grad}(z + P_2^*)|} \tag{18}$$

When the porosity is high or the specific surface is small, then the linear flow equation, equation (16), will produce higher values of flow in comparison to a more precise formula, equation (18). Let  $\bar{q}_{20}$  be the value of flow with linear filtration. Then the ratio of the absolute value of nonlinear flow to linear flow can be derived:

$$|\bar{q}_2| / |\bar{q}_{20}| = \left[ \sqrt{1 + 2q_{20} / u_g} - 1 \right] / (q_{20} / u_g) \tag{19}$$

The values of these ratios are presented in Table 1.

**Table 1** The ratio of nonlinear flow to linear flow for different values of  $q_{20}/u_g$ .

$q_{20}/u_g$	0.1	0.2	0.4	0.6	0.8	1.0	1.5
$q_2/q_{20}$	0.954	0.916	0.854	0.805	0.766	0.732	0.667

(b) **Pores are not completely filled by water** ( $\alpha_3 > 1$ ). Introduce the quantity

$$\varphi = \frac{\alpha_2}{(1 - \alpha_1)}, \quad 0 \leq \varphi \leq 1 \tag{20}$$

called the moisture saturation. According to Aravin & Numerov (1953), Bezborodov & Khalbaeva (1985), Denisov (1968) and equation (3), we have:

$$\delta_2 = k \frac{(1 - \alpha_1)\varphi}{\beta_c + \beta_{23}} = \delta_n \frac{\varphi}{\left[ 1 + (1 - \varphi)^{\frac{k-1}{k}} \right]} = \delta_n \frac{\varphi}{\Omega} \tag{21}$$

Water pressure in the unsaturated zone is negative and determined by the suction potential  $\Psi_2$  (Denisov, 1968, 1978; Nerpin & Chudnovskiy, 1967):

$$P_2 = -\gamma_2 \Psi_2 \tag{22}$$

This potential is the sum of three potentials: the frame potential (attributed to the attraction of water by the soil matrix), the meniscus potential (attributed to curvature of water surface in unsaturated zone) and the osmotic potential arising from the solute concentration of the soil water.

Reasoning similarly and using equations(5)–(18), the expression for water movement in a porous medium when the pores are not completely filled by water can be derived:

$$\bar{q}_2 = -u_g \Omega \left[ \sqrt{1 + 2 \frac{K_\phi}{u_g \Omega} \frac{\Phi^3}{\Omega} |\text{grad}(z - \Psi_2)|} - 1 \right] \frac{\text{grad}(z - \Psi_2)}{|\text{grad}(z - \Psi_2)|} \quad (23)$$

For a negligible value of the second term in equation (23), (the term under the root):

$$\bar{q}_2 = -K_\phi \frac{\Phi^3}{\Omega} \text{grad}(z - \Psi_2) = -K_B \text{grad}(z - \Psi_2) \quad (24)$$

Here  $K_B$  is the unsaturated hydraulic conductivity coefficient.

## MODEL VERIFICATION

In conclusion let us find out how earthworms influence the values of the saturated and unsaturated hydraulic conductivity. Let  $\alpha_{10}$  and  $\beta_{n0}$  be the relative volume of the soil matrix and its specific surface in the absence of earthworms, respectively. Let  $\chi$  be the length of the burrows of earthworms in the unit of soil volume and  $\delta_\chi$  be the radius of the cross-section of the earthworm burrows. The volume of the soil matrix is therefore reduced by  $\pi \delta_\chi^2 \chi \beta_{n0} \Delta V$ . The relative volume of the soil matrix,  $\alpha_1$ , then equals:

$$\alpha_1 = \frac{\Delta V_{10} - \pi \delta_\chi^2 \chi \alpha_{10} \Delta V}{\Delta V} = (1 - \pi \delta_\chi^2 \chi) \alpha_{10} \quad (25)$$

and the specific surface  $\beta_n$  equals:

$$\beta_n = \frac{\Delta S_{10} - \pi \delta_\chi^2 \chi \beta_{c0} \Delta V}{\Delta V} = (1 - \pi \delta_\chi^2 \chi) \beta_{c0} \quad (26)$$

Substituting equations (25) and (26) for the equation of the saturated hydraulic conductivity coefficient (17), and after some simple re-arrangement we obtain

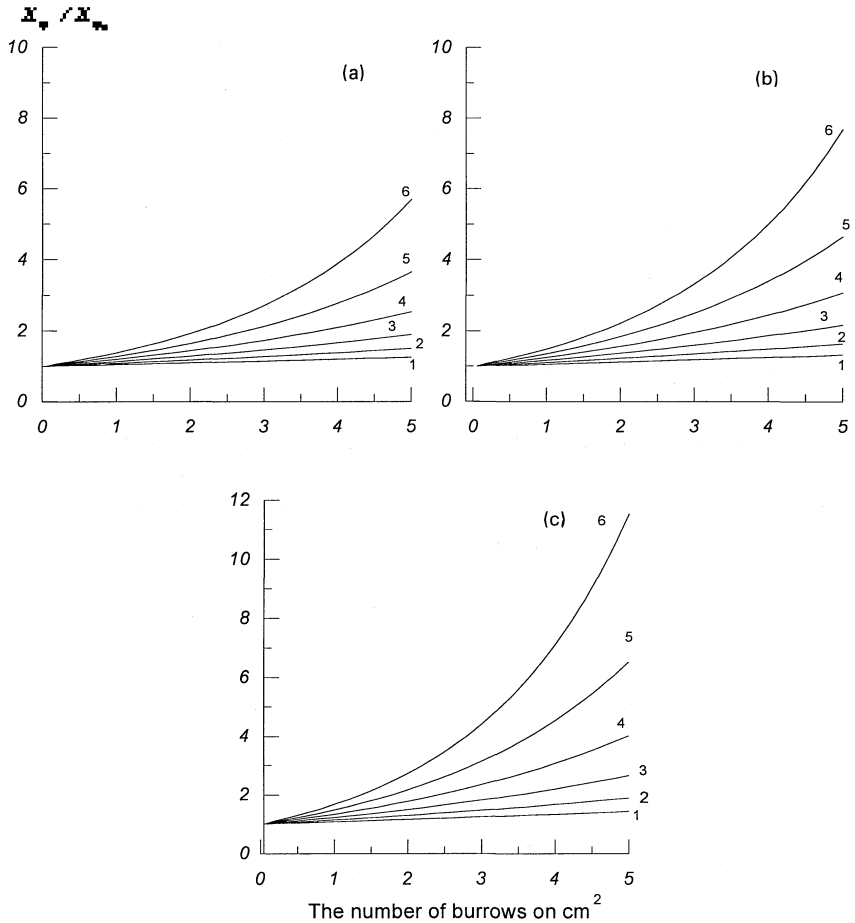
$$K_\phi = K_{\phi 0} \frac{\left[ 1 + \frac{\pi \delta_\chi^2 \chi \alpha_{10}}{(1 - \alpha_{10})} \right]^3}{(1 - \pi \delta_\chi^2 \chi)^2} \quad (27)$$

where  $K_{\phi 0}$  is the hydraulic conductivity coefficient when the burrows of earthworms are absent.

In practice it is very difficult to determine the length of the earthworm burrows. In practice experimenters count the number of holes on the surface per unit of area,  $j$ . An approximate relationship between  $\chi$  and  $j$  can be expressed as follows:

$$\chi = 4j \sqrt{1 + \frac{1}{4\Delta z^2 j}} \quad (28)$$

where  $\Delta z$  is the depth of the plough layer.



**Fig. 1** The change of hydraulic conductivity coefficient of soil depending on the number of earthworm burrows, the cross sectional-radius of earthworm burrows and the porosity of the soil. Porosity: (a) 60%; (b) 50%; (c) 40%. Radius (cm): (1) 0.06; (2) 0.10; (3) 0.14; (4) 0.16; (5) 0.18; (6) 0.20.

The order of the values of  $\beta_n$ , equation (26) and  $K_{\phi}$ , equation (27) coincide with values measured in field conditions with different values of  $j$ ,  $\delta_{\chi}$  and  $\alpha_{10}$ .

The values of  $K_{\phi} / K_{\phi_0}$  for different  $j$  and  $\delta_{\chi}$  when the porosity equals 0.60; 0.50 and 0.40 (60%, 50% and 40%) are given in Fig. 1.

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