Physically-based mathematical formulation for hillslope-scale prediction of erosion in ungauged basins

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Abstract An explicit finite difference scheme for erosion and sediment transport on upland areas of a watershed is derived. The derivation is based on the unsteady state one-dimensional sediment continuity and momentum equations, simplified with the kinematic wave approximation. The derivation ends up with a linear partial differential equation. Upland erosion is thought of as sheet erosion incorporating the effects of rainfall and runoff by way of non-physical calibration parameters. Calibration of these parameters is of great importance for ungauged basins where data do not exist. A finite difference scheme is chosen to solve the resulting equation, together with appropriate boundary and initial conditions. A hypothetical data set was used to evaluate the applicability of the model developed in the study. The performance of the model at the hillslope scale indicates that it has potential for application at the watershed-scale.

Key words erosion; hillslope; mathematical model; rainfall erosion; runoff erosion; ungauged basin

NOTATION

C Chezy roughness coefficient [$L^{1/2}$ T^{-1}]
C_s Volumetric sediment concentration [L^3 L^{-3}]
E Erosion [ML^{-2} T^{-1}]
E_r Rainfall erosion [ML^{-2} T^{-1}]
E_f Runoff erosion [ML^{-2} T^{-1}]
h Flow depth [L]
q Unit width discharge [L^2 T^{-1}]
q_s Unit sediment discharge [ML^{-1} T^{-1}]
R Rainfall intensity [LT^{-1}]
S Topographical slope [–]
t Time [T]
T_c Transport capacity of flow [ML^{-1} T^{-1}]
v Velocity [LT^{-1}]
x Downslope distance [L]
α Coefficient ($=C\sqrt{S}$) [$L^{1/2}$ T^{-1}]

β Exponent
χ Dimensional rainfall erosion coefficient
δ Rainfall erosion exponent
ε Exponent
γ Specific weight of water [$ML^{-2} T^{-2}$]
η Dimensional coefficient
ρ_s Specific mass of sediment [ML^{-3}]
σ Transfer rate coefficient [L^{-1}]
τ Flow shear stress [$ML^{-1} T^{-2}$]
τ_c Critical shear stress [$ML^{-1} T^{-2}$]
Δt Time increment [T]
Δx Space increment [L]

INTRODUCTION

Erosion is a process of detachment and transportation of soil materials by different agents such as wind, rainfall and runoff (Foster & Meyer, 1972). It is also defined as the loosening, or dissolving, and removal of earth or rock materials from any part of the Earth's surface (ASCE Task Committee, 1970). Erosion on planar hillslopes with no rill development is rainfall-dominated, whereas erosion in channels is mostly controlled by runoff. Rainfall detaches sediment particles from their original locations on the Earth's surface due to raindrop impact. Sediment concentration is the highest at the beginning of rainfall, and decreases until a steady state is reached. As mean water depth increases, sediment concentration decreases. The decrease in soil loss at greater flow depths can be attributed to the increasing protection of the soil surface from raindrop impact as the flow depth on the surface increases (Singh & Prasad, 1982).

The upland area of a watershed is that part of the landscape where runoff is considered as overland flow in hydrological analysis. Although an upland area has a micro-topographical structure with rills and interrill areas, the derivation performed in this study does not take this structure into account and considers erosion on upland areas as sheet erosion comprised of rill and interrill erosion. Flow usually has no effect on the detachment of soil in upland areas since shear stress is low because of the very low flow depth. Rainfall is assumed to be the unique agent to detach the soil. Raindrop impact and flow combine to transport the detached soil particles to channels in the lower parts of the watershed.

In the following sections, a hydrological analysis of flow over a hillslope is first performed, then a sediment transport equation is derived at the hillslope scale. The erosion term is described in detail in the following section. A finite difference scheme-based numerical solution of the formulation is then given, together with an application to a hypothetical data set.

HYDROLOGICAL FORMULATION

The equation governing one-dimensional unsteady flow on an impervious slope is given by:

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=R \tag{1}$$

where h is depth of water over the slope [L], t is time [T], q is unit width discharge of flow [L^2 T^{-1}], x is downslope distance [L], and R is rainfall intensity [LT^{-1}] (Foster, 1982). The unit width discharge is given by:

$$q = vh \tag{2}$$

where v is average velocity of flow [LT^{-1}]. A slope over which water flows can be considered to be a wide channel. Using the Chezy equation for surface roughness:

$$v=\alpha\sqrt{h} \tag{3}$$

with $\alpha = CS^{1/2}$ where C is Chezy roughness coefficient [$L^{1/2}$ T^{-1}] and S is topographical slope [–], leads to:

$$\frac{\partial h}{\partial t}+\frac{\partial\left(\alpha h^{3/2}\right)}{\partial x}=R \tag{4}$$

If we write the exponent as β (= 3/2), equation (4) can also be written as:

$$\frac{\partial h}{\partial t}+\alpha\beta h^{\beta-1}\frac{\partial h}{\partial x}=R \tag{5}$$

However, taking β = 1 can also be considered, resulting in:

$$\frac{\partial h}{\partial t}+\alpha\frac{\partial h}{\partial x}=R \tag{6}$$

This first order linear partial differential equation was proposed by Giakoumakis & Tsakiris (2001) who compared its performance in simulating the flow over infiltrating surfaces to the results of laboratory experiments.

SEDIMENT TRANSPORT

With regard to sediment yield, a watershed can be divided into two main parts: upland and lowland (Bennett, 1974). In the upland area of the watershed, runoff is considered to be overland flow in hydrological analysis. Erosion due to overland flow takes place in the upland part of the watershed. The upland area collects the runoff and transports it to channels. Although there is a tremendous variability (rills and interrill areas), the upland area is considered a smooth plane in most previous studies, including KINEROS (Woolhiser *et al.*, 1990) and SHESED (Wicks, 1988). Recently, interaction of rills and interrill areas was considered in hydrological studies (Tayfur & Kavvas, 1994) and erosion and sediment transport studies (Govindaraju & Kavvas, 1992; Morgan *et al.*, 1998; Aksoy & Kavvas, 2001). In this study, no distinction is made between rill and interrill areas. Therefore, the upland area is considered to be a smooth plane.

The one-dimensional continuity equation for sediment can be derived as:

$$\frac{\partial\left(hC_s\right)}{\partial t}+\frac{\partial\left(qC_s\right)}{\partial x}=\frac{E}{\rho_s} \tag{7}$$

where C_s is volumetric sediment concentration [L^3L^{-3}], ρ_s is specific mass of sediment [ML^{-3}], and E is erosion rate [ML^2T^{-1}], which is the sum of erosion by rainfall and overland flow as explained in detail in the following section.

$$\frac{\partial(hC_s)}{\partial t}+\alpha\frac{\partial(hC_s)}{\partial x}=\frac{E}{\rho_s} \tag{8}$$

is obtained following the assumptions and simplifications made in the hydrological analysis. It should be noted that h, C_s and E in equations (6) and (8) are functions of both space and time.

EROSION

Erosion occurs in the upland and lowland parts (channels) of the watershed. In this study, only upland erosion and sediment transport were formulated. Erosion taking place in the channel itself is out of the scope of this study.

Upland erosion starts with the detachment of sediment particles. Once a sediment particle is set in motion, it is much easier for the flow to transport it in the direction of flow because a lower shear stress is required. Many factors influence the erosional processes so that many parameters need to be used in calculating erosion. An important task in erosion and sediment transport studies is the calibration and validation of those parameters. In this study, the erosion term (E) in equation (8) is equal to the sum of erosion by rainfall (E_r) and overland flow (E_f).

Erosion by rainfall

Erosion by rainfall is usually related to the rainfall intensity (R) as:

$$E_r=\chi R^{\delta} \tag{9}$$

where χ and δ are parameters to be determined in the calibration stage of the model or to be set at some values known from the literature. χ is the soil detachability coeff-icient. It has a dimension that depends on the value of δ, which usually equals 1 or 2. It should be noted that rainfall intensity is constant throughout the storm and uniform over the entire upland area. χ can also be considered a function of rainfall intensity as in SHESED (Wicks, 1988), where the coefficient was given for four rainfall intensity class intervals. In SHESED, not only χ but also δ was considered dependent upon rainfall intensity. Higher rainfall intensities result in lower values of χ and δ.

Erosion by overland flow

Net shear stress is defined as the difference between the shear stress produced by the flow and the critical shear stress, which is the minimum shear stress required for sediment particles to be detached. Erosion occurs as long as the net shear stress on the soil surface is greater than zero. Eroded sediment is transported as long as the sediment load is smaller than the transport capacity of the flow. Based upon this concept, the following basic equation was used for erosion by overland flow:

$$E_f = \sigma(T_c - q_s) \tag{10}$$

in which σ is the transfer rate coefficient or a first order reaction coefficient for deposition [L^{-1}] (Foster, 1982). In equation (10), T_c is the transport capacity of the flow [$ML^{-1}T^{-1}$] and q_s is the unit sediment discharge [$ML^{-1}T^{-1}$]. Note that E_f is erosion from a unit area of hillslope [$ML^{-2}T^{-1}$].

The sediment discharge in equation (10) is defined as:

$$q_s = \rho_s C_s q \tag{11}$$

where ρ_s, C_s and q are as previously defined. The transport capacity of the flow is usually given by an equation that incorporates net shear stress, as:

$$T_c = \eta(\tau - \tau_c)^{\varepsilon} \tag{12}$$

where η is a dimensional coefficient and ε is an exponent. τ and τ_c are the shear stress [$ML^{-1}T^{-2}$] and its critical value, respectively. The critical shear stress can be thought of as the resistance of soil against erosion, so that a higher critical shear stress implies a less erodible soil. The shear stress exerted by the flow is given by:

$$\tau = \gamma h S \tag{13}$$

where γ is the specific weight of water [$ML^{-2}T^{-2}$], h is flow depth [L] and S is topographical slope [–]. This equation is valid when the cross-section is assumed to be a wide rectangle, which is a reasonable approximation for overland flow when the hydraulic gradient can be equated to the topographical slope of the upland area (kinematic wave approximation).

Critical shear stress is a parameter in equation (12) that should be determined. However, it can be assumed that there are always fine particles of sediment detached by the action of wind and other elements between storm events. These sediment particles are transported by overland flow as soon as the rainfall intensity exceeds the infiltration capacity of the soil surface, without any resistance to removal (Lopes, 1987). In this study, the critical shear stress, τ_c, was therefore set to zero for flow over upland areas.

There are three parameters in equations (10) and (12), σ, η and ε, that need to be calibrated. When the critical shear stress is taken into account, additional parameters, depending upon the chosen equation, will arise. The erosion term E in equation (8) becomes:

$$E = \chi R^{\delta} + \sigma\left[\eta(\gamma h S)^{\varepsilon} - \rho_s C_s \alpha h\right] \tag{14}$$

Initial and boundary values of overland flow depth and sediment concentration were set to zero. The sediment concentration then becomes an explicit function of overland flow depth at the current and previous time and space steps, and of sediment concentration at the previous time and space steps.

$$C_s(x;t) = func\{h(x;t-\Delta t), h(x-\Delta x;t), h(x;t), C_s(x;t-\Delta t), C_s(x-\Delta x;t)\} \tag{15}$$

Δx and Δt in equation (15) are space and time increments, respectively. Overland flow depth is already known from the hydrological part of the model, whereas sediment concentration of the previous time and space steps is taken from the solution of the numerical algorithm.

APPLICABILITY OF THE MODEL

A hypothetical data set was chosen to test the model. A limited number of references was found to contain values of coefficients and exponents for the rainfall and runoff equations (Foster, 1982). Table 1 is a summary of the hypothetical data set used in the study. The objective of this study was to see if the proposed algorithm could be used for erosion and sediment transport studies. Once the applicability of the formulation is confirmed, it can be extended to infiltrating surfaces at the hillslope and watershed scales. The inputs required by the model are rainfall data and the calibration parameters for the erosion part of the model. The rainfall data can be measured or estimated. The calibration parameters can be set to some values known from the literature.

Table 1 Hypothetical data set used in the study.

Rainfall intensity (R)	20 mm h^{-1}
Chezy coefficient (C)	15 $m^{1/2}s$
Slope (S)	0.05
Rainfall duration	15 min
Length of slope	10 m
Rainfall erosion coefficient (χ)	1
Rainfall erosion exponent (δ)	2
Transfer rate coefficient (σ)	1.3 m^{-1}
Specific mass of sediment (ρ_s)	2700 kg m^{-3}
Coefficient (η)	0.06
Exponent (ε)	1

Results of the application are presented in Figs 1 and 2, which show the generated hydrograph and sedigraph, respectively. The algorithm was run for 20 min, which is 5 min longer than the rainfall duration. It is seen that runoff increases and then stabilizes during the rainfall and, as expected, decreases suddenly after rainfall ceased. A similar sedigraph was obtained with increasing sediment discharge during the storm and decreasing sediment discharge after the rainfall had stopped. Figures 1 and 2 show the applicability of the formulation.

CONCLUSION

Based on the hypothetical results obtained in this study, it can be concluded that the proposed model is of potential use for hillslope scale hydrological analysis and erosion and sediment transport studies. The model should be tested with experimental field and laboratory data so that the parameters required by the model can be calibrated. The

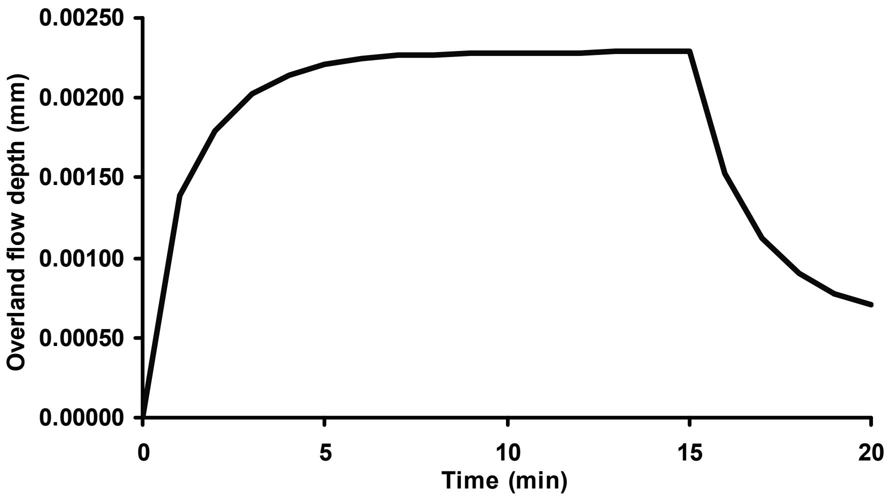

Fig. 1 Simulated hydrograph at the bottom of the hillslope.

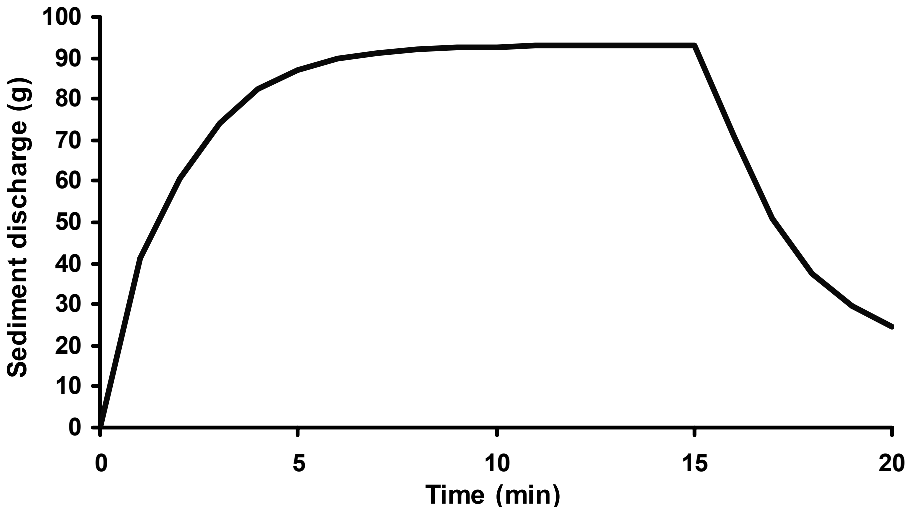

Fig. 2 Simulated sedigraph at the bottom of the hillslope.

only requirement other than the calibration parameters is the rainfall. Data from basins with similar physical characteristics to those of the ungauged basin to be investigated can be used for setting the values of the calibration parameters.

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REFERENCES

Aksoy, H. & Kavvas, M. L. (2001) A physically-based erosion and sediment transport component for watershed hydrologic models. Hydrologic Research Laboratory, Department of Civil and Environmental Engineering, University of California, Davis, California, USA.

ASCE Task Committee (1970) Sediment sources and sediment yields. *J. Hydraul. Div. ASCE* **96**(HY6), 1283–1329.

Bennett, J. P. (1974) Concepts of mathematical modeling of sediment yield. *Water Resour. Res.* **10**, 485–492.

Foster, G. R. (1982) Modeling the erosion process. In: *Hydrologic Modeling of Small Watersheds* (ed. by C. T. Haan, H. P. Johnson & D. L. Brakensiek), 297–380. ASAE Monograph no. 5. ASAE, St. Joseph, Michigan, USA.

Foster, G. R. & Meyer, L. D. (1972) A closed-form soil erosion equation for upland areas. In: *Sedimentation (Symposium to Honor Professor H. A. Einstein)* (ed. by H. W. Shen), 12.1–12.19. Colorado State University, Fort Collins, Colorado, USA.

Giakoumakis, S. & Tsakiris, G. (2001) Experimental validation of a linearized kinematic wave equation for micro-catchment water harvesting design. *Water Resour. Manage.* **15**, 235–246.

Govindaraju, R. S. & Kavvas, M. L. (1992) Characterization of the rill geometry over straight hillslopes through spatial scales. *J. Hydrol.* **130**, 339–365.

Lopes, V. L. (1987) A numerical model of watershed erosion and sediment yield. PhD Thesis, University of Arizona, Tucson, Arizona, USA.

Morgan, R. P. C., Quinton, J. N., Smith, R. E., Govers, G., Poesen, J. W. A., Auerswald, K., Chisci, G., Torri, D. & Styczen, M. E. (1998) The European soil erosion model (EUROSEM): A dynamic approach for predicting sediment transport from fields and small catchments. *Earth Surf. Processes Landf.* **23**, 527–544.

Singh, V. P. & Prasad, S. N. (1982) Explicit solution to kinematic equations for erosion on an infiltrating plane. In: *Modeling Components of Hydrologic Cycle* (ed. by V. P. Singh), 515–538. Water Resources Research, Littleton, Colorado, USA.

Tayfur, G. & Kavvas, M. L. (1994) Spatially averaged conservation equations for interacting rill-interrill area overland flows. *J. Hydraul. Eng. ASCE* **120**(12)**,** 1426–1448.

Wicks, J. M. (1988) Physically-based mathematical modelling of catchment sediment yield. PhD Thesis, Department of Civil Engineering, University of Newcastle Upon Tyne, UK.

Woolhiser, D. A., Smith, R. E. & Goodrich, D. C. (1990) *KINEROS, a Kinematic Runoff and Erosion Model.* Documentation and user manual, USDA, ARS-77.