# Estimating unsaturated soil hydraulic parameters using a generalized Richards equation

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**Abstract** Richards equation is widely used as the basis for simulations of water transport in soils. For infiltration into horizontal soil columns, Richards equation predicts that the water content profile is a unique function of the Boltzmann variable (distance)/(time)$^q$, where $q = 0.5$. However, a number of experiments have found that $q$ is significantly less than 0.5. Scaling with $q < 0.5$ is consistent with a generalized Richards equation that uses a fractional time derivative of the water content. In this paper we consider a generalized Richards equation that incorporates the unsaturated hydraulic conductivity function $k(h)$ of van Genuchten (1980). A new method is proposed for estimating the van Genuchten parameters $\alpha$ and $n$. Estimates of $\alpha$ and $n$ are expressed as closed form equations that are functions of other parameters such as the length of wetted zone, the sorptivity, and the saturated hydraulic conductivity.

**Key words** generalized Richards equation; unsaturated hydraulic conductivity function

## INTRODUCTION

The hydraulic conductivity of a soil determines the rate at which water can flow in the soil and consequently the rate at which suspended or dissolved materials can be transported. Predicting flow and transport requires accurate knowledge of the hydraulic conductivity. However, determining the unsaturated hydraulic conductivity is difficult and complex. Hydraulic conductivity is a highly nonlinear function of pressure head or water content, and field and laboratory measurements are difficult and time-consuming. Alternative methods for estimating or calculating the conductivity are difficult to validate.

Methods for determining the conductivity generally use some mathematical form for the conductivity function. However, no universal equation exists for the unsaturated hydraulic conductivity. In 1980, van Genuchten put forward an equation that specifies the conductivity as a function of soil water pressure head:

$$k(h) = \frac{k_s\left[1-\dfrac{(\alpha\,|\,h\,|)^{n-1}}{\left(1+(\alpha\,|\,h\,|)^n\right)^{1-1/n}}\right]^2}{\sqrt{\left(1+(\alpha\,|\,h\,|)^n\right)^{1-1/n}}} \tag{1}$$

where $k(h)$ is the hydraulic conductivity (LT$^{-1}$), $k_s$ is the saturated conductivity (LT$^{-1}$), and $\alpha$ (L$^{-1}$) and $n$ (-) are adjustable parameters. It is difficult to estimate $\alpha$ and $n$. In this paper, a new method for estimating $\alpha$ and $n$ is presented. The method is based on a generalized Richards equation. Estimates of $\alpha$ and $n$ are expressed as closed form equations that are functions of the characteristic length of the wetted zone, the sorptivity, and the saturated hydraulic conductivity.

## GENERALIZED RICHARDS EQUATION

The Richards equation for one dimensional (1-D) horizontal flow may be written as:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[k(h)\frac{\partial h}{\partial x}\right]$$

where $\theta$ is the volumetric water content ($L^3\ L^{-3}$) and $h$ is the pressure head (L). The Richards equation was obtained originally in 1931 by combining the equation of continuity with the empirically based "Buckingham-Darcy" flux law (Swartzendruber, 1968). Later, Bhattacharya *et al.* (1976) showed that the Richards equation can be derived from physically-based molecular principles. The derivation is based on the assumption that water moves in a Brownian motion in the form of quasi-molecules, an assumption that has also been instrumental in deriving solute diffusion and convective–dispersive transport equations (Bhatacharya & Gupta, 1990).

Both the horizontal water transport equation and the solute diffusion equation conform to Boltzmann scaling, wherein water or solute profiles are unique functions of the Boltzmann variable (distance)/(time)$^q$, with $q = 0.5$ (Hillel, 1980). However, both water flow and solute transport in porous media have been shown to deviate from Boltzmann scaling and follow a more general scaling law with $q \neq 0.5$ (Neuman, 1990; Hatano & Hatano, 1998; Haggerty *et al.*, 2000; Pachepsky *et al.*, 2001, 2003). A physical model that is consistent with this more general scaling consists of particles undergoing non-Brownian motion due to the structure of, or interactions with, the porous media (Metzler & Klafter, 2000; Pachepsky *et al.*, 2003). Scaling with $q > 0.5$ occurs when particles undergo Levy motion, which is similar to Brownian motion, except that relatively large transitions occur more often, such as may be caused by macropores or fractures. Scaling with $q < 0.5$ occurs when particles remain motionless for extended periods of time. Pachepsky *et al.* (2003) noted that this type of behaviour has long been known in soil wetting, with Nielsen *et al.* (1962) suggesting in 1962 that $q < 0.5$ may occur when the infiltration front experiences sporadic periods of immobility and thus progresses in "jerky movements". A physical model consistent with these dynamics is that of water particles being randomly trapped and having a power law distribution of waiting times. Pachepsky *et al.* (2003) proposed a generalized Richards equation based on this concept.

Particles with a power law distribution of waiting times have an infinite mean waiting time, and it has been suggested that the transport of an ensemble of such particles can be simulated using a fractional time derivative (Meerschaert *et al.*, 2002; Pachepsky *et al.*, 2003). Thus the generalized Richards equation is similar to the original except that the fractional time derivative of water content is used:

$$\frac{\partial^\gamma \theta}{\partial t^\gamma} = \frac{\partial}{\partial x}\left[k(h)\frac{\partial h}{\partial x}\right] \tag{2}$$

where $\gamma(\leq 1)$ is the order of the fractional derivative. Equation (2) reduces to the classical Richards equation when $\gamma = 1$. The fractional derivative is defined by:

$$\frac{\partial^\gamma \theta}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t \frac{\theta(\tau,x)}{(t-\tau)^\gamma}d\tau$$

The initial and boundary conditions for horizontal infiltration into a semi-infinite column are:

$$h(x,0) = h_0,\ h(0,t) = 0,\ h(\infty,t) = h_0$$

where $h_0$ is initial pressure head.

Application of the Boltzmann transform

$$\xi = \frac{x}{t^{\gamma/2}} \tag{3}$$

yields

$$\frac{\partial^\gamma \theta}{\partial t^\gamma} = \frac{d\theta}{d\xi}\frac{\partial^\gamma \xi}{\partial t^\gamma} = \frac{d\theta}{d\xi}x\frac{d^\gamma\left(t^{-\gamma/2}\right)}{dt^\gamma}$$

$$= \frac{d\theta}{d\xi}x\frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}t^{-\gamma/2-\gamma}$$

$$=\frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}\frac{x}{t^{\gamma/2+\gamma}}\frac{d\theta}{d\xi}$$

and thus:

$$\frac{\partial}{\partial x}\left[k(h)\frac{\partial h}{\partial x}\right]=\frac{1}{t^{\gamma}}\frac{d}{d\xi}\left[k(h)\frac{dh}{d\xi}\right].$$

Equation (2) becomes:

$$\frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}\xi\frac{d\theta}{d\xi}=\frac{d}{d\xi}\left[k(h)\frac{dh}{d\xi}\right] \tag{4}$$

with boundary conditions:

$$h(0)=0,\ h(\infty)=h_0 \tag{5}$$

## HYDRAULIC CONDUCTIVITY

In this section, we incorporate the van Genuchten (1980) soil hydraulic conductivity function into the generalized Richards equation and derive closed form expressions useful for estimating van Genuchen parameters. Mualem (1976) presented a statistical pore-size distribution model that related the soil water retention curve to the hydraulic conductivity function. Van Genuchten (1980) proposed a new functional form for the water retention curve and used the model of Mualem (1976) to derive a new expression for the conductivity, equation (1). The water retention function from Van Genuchten (1980) is:

$$\theta(h)=\begin{cases}\theta_r+\dfrac{\theta_s-\theta_r}{\left[1+|\alpha h|^n\right]^{1-1/n}}, & h<0\\ \theta_s, & h\geq 0\end{cases} \tag{6}$$

where $\theta_s$ is the saturated soil water content ($L^3\ L^{-3}$), $\theta_r$ is the residual soil water content ($L^3\ L^{-3}$), and $\alpha$ and $n$ are the adjustable parameters discussed above in relation to equation (1). Considerable effort has been directed at developing procedures for estimating values for $\alpha$ and $n$ (e.g. Van Genuchten *et al.*, 1992). In the following, we propose a new method for estimating $\alpha$ and $n$ using the generalized Richards equation and present closed form expressions for the soil hydraulic parameters.

We start by expressing the pressure head as a Maclaurin series:

$$h(\xi)=a_0+a_1\xi+a_2\xi^2+\cdots+\cdots$$

Because $h(0)=0$, we have $a_0=0$ and

$$h(\xi)=a_1\xi+a_2\xi^2+\cdots+\cdots$$

where $a_1$ is a negative constant. For convenience, we denote $b_1=a_1$. If $\xi$ is small enough, then $h(\xi)\approx -b_1\xi$. However, for small $\alpha|h|^n$, we have:

$$\left[1+(\alpha\,|\,h\,|)^n\right]^{1-1/n}\approx 1+m(\alpha\,|\,h\,|)^n$$

So,

$$\frac{\theta-\theta_r}{\theta_s-\theta_r}=1-m(\alpha b_1\xi)^n \tag{7}$$

where $m = 1 - \frac{1}{n}$. At a characteristic length $\xi = d$ (the depth of the leading edge of the wetting front), the water content is defined $\theta(d) = \theta_0$. Equation (7) then becomes:

$$mb_1^n = \frac{\theta_s - \theta_0}{(\theta_s - \theta_r)(\alpha d)^n} \tag{8}$$

Combining equations (7) and (8) gives:

$$\theta(\xi) = \begin{cases} \theta_s - (\theta_s - \theta_0)\left(\frac{\xi}{d}\right)^n, & 0 < \xi < d \\ \theta_0, & d \le \xi < \infty \end{cases}$$

This water content profile is consistent with the function proposed by Philip (1955) and the experimental data of Shao (2000).

We now consider integrating equation (4). Integration of the right-hand side of equation (4) gives:

$$\int_0^\infty \frac{d}{d\xi}\left[k(h)\frac{dh}{d\xi}\right] = k(h)\frac{dh}{d\xi}\Big|_0^\infty = b_1 k_s$$

This result follows from $\left.\frac{dh}{d\xi}\right|_{\xi=\infty} = 0$, $\left.\frac{dh}{d\xi}\right|_{\xi=0} = -b_1$, $h(0) = 0$, and $k(0) = k_s$. For the left-hand side, we recall the expression given above for $\theta(\xi)$ and obtain

$$\int_0^\infty \frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}\xi\frac{d\theta}{d\xi}d\xi = \frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}\int_0^d \xi\frac{d\theta}{d\xi}d\xi$$

$$= -\frac{\Gamma(1-\gamma/2)}{\Gamma(1-3\gamma/2)}\frac{n(\theta_s-\theta_0)d}{(n+1)}$$

Thus if we integrate equation (4) and make the substitution (see equation (8)):

$$b_1 = \frac{1}{\alpha d}\left[\frac{\theta_s - \theta_i}{m(\theta_s - \theta_r)}\right]^{\frac{1}{n}} \tag{9}$$

we obtain an expression for $\alpha$

$$\alpha = -\frac{\Gamma(1-3\gamma/2)}{\Gamma(1-\gamma/2)}\frac{(n+1)k_s}{n(\theta_s-\theta_0)d^2}\left[\frac{\theta_s-\theta_0}{m(\theta_s-\theta_r)}\right]^{\frac{1}{n}} \tag{10}$$

Next, we seek an expression for the parameter $n$. At $x = 0$, the soil water flux is:

$$q = -\left(k(h)\frac{\partial h}{\partial x}\right)\Big|_{x=0}.$$

Since $\left.\frac{\partial h}{\partial x}\right|_{x=0} = \left(\frac{\partial h}{\partial \xi}\right)_{x=0}\left(\frac{\partial \xi}{\partial x}\right)_{x=0} = -b_1 t^{-\gamma/2}$ and $k(h)|_{x=0} = k_s$, the flux may also be expressed:

$$q|_{x=0} = k_s b_1 t^{-\frac{\gamma}{2}} \tag{11}$$

For horizontal infiltration, the Phillip infiltration formula has the form:

$$q = \frac{S}{2} t^{-\frac{\gamma}{2}} \tag{12}$$

where $S = \int_{\theta_{dry}}^{\theta_{wet}} \xi d\theta$ is the sorptivity. Combining equations (9), (11), and (12) results in an expression for $\alpha$:

$$\alpha = \frac{2k_s}{dS} \left[ \frac{\theta_s - \theta_0}{m(\theta_s - \theta_r)} \right]^{\frac{1}{n}} \tag{13}$$

Substituting equation (13) into equation (10) yields an expression for $n$ that does not contain $\alpha$:

$$n = \frac{S\Gamma(1-3\gamma/2)}{-S\Gamma(1-3\gamma/2) - 2\Gamma(1-\gamma/2)(\theta_s - \theta_0)d}. \tag{14}$$

## DISCUSSION AND CONCLUSION

Equations (13) and (14) are closed-form expressions for $\alpha$ and $n$ in terms of the parameters $k_s$, $S$, $d$, $\theta_s$, $\theta_0$, $\theta_r$ and $\gamma$. When these parameter values are available, equation (14) can be used to calculate $n$ and then equation (13) can be used to calculate $\alpha$. Measurements of horizontal water adsorption can be used to determine $\gamma$ and $S$ (equation (12)). The parameters $k_s$, $\theta_s$, $\theta_0$, $\theta_r$, are relatively easy to estimate or measure, whereas the characteristic length $d$ can be estimated as a distance between the surface and the wetting front. The method developed in this paper is a quick and inexpensive approach to estimating hydraulic parameters.

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## REFERENCES

Bhatacharya, R. & Gupta, V. K. (1990) Application of the central limit theorem to solute transport in saturated porous media: from kinetic to field scales. In: *Dynamics of Fluids in Hierarchical Porous Media* (ed. by J. H. Cushman), 97–124. Academic Press, New York, USA.

Bhattacharya, R. N., Gupta, V. K. & Sposito, G. (1976) A Markovian stochastic basis for the transport of water through unsaturated soil. *Soil Sci. Soc. Am. J.* **40**(2), 465–467.

Haggerty, R., McKenna, S. A. & Meigs, L. C. (2000) On the late-time behavior of tracer test breakthrough curves. *Water Resour. Res.* **36**(12), 3467–3480.

Hatano, Y. & Hatano, N. (1998) Dispersive transport of ions in column experiments: an explanation of long-tailed profiles. *Water Resour. Res.* **34**(3), 1027–1041.

Hillel, D. (1980) *Fundamentals of Soil Physics*. Academic Press, San Diego, USA.

Meerschaert, M. M., Benson, D., Scheffler, H.-P. & Baeumer, B. (2002) Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E.* **65**, 041103.

Metzler, R. & Klafter, J. (2000) The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**(ERI), 1–77.

Mualem, Y. (1976) A new model for predicting the hydraulic conductivity of unsaturated porous media. *Water Resour. Res.* **12**(2), 593–622.

Neuman, S.P. (1990) Universal scaling of hydraulic conductivities and dispersivities in geologic media. *Water Resour. Res.* **26**, 1749–1758.

Nielsen, D. R., Biggar, J. W. & Davidson, J. M. (1962) Experimental considerations of diffusion analysis in unsaturated flow problem. *Soil Sci. Soc. Am. Proc.* **26**, 107–111.

Pachepsky, Ya., Benson, D. & Timlin, D. (2001) Transport of water and solutes in soil as in fractal media. SSSA Special Publication, Soil Science Society of America, Madison, Wisconsin, USA, **56**, 51–75.

Pachepsky, Y., Timlin, D. & Rawls, W. (2003) Generalized Richards' equation to simulate water transport in unsaturated soils. *J. Hydrol.* **272**, 3–13.

Philip, J. R. (1955) Numerical solution of equations of the diffusion type with diffusivity concentration-dependent. *Transactions of the Faraday Society* **51**, 885–892.

Shao, M., Wang, Q. & Robert, H. (2000) A simple infiltration method for estimating soil hydraulic properties of unsaturated soils, theoretical analysis. *Acta Pedologica Sinica* **37**(1), 1–8.

Swartzendruber, D. (1968) The applicability of Darcy's law. *Soil Sci. So. of Am. Proc.* **32**, 11–18.

van Genuchten M. Th. (1980) A closed form equation for predicting the hydraulic conductivity of unsaturated soils. *Soil Sci. Soc. Am. J.* **44**, 892–898.

van Genuchten, M.Th., Leij, F.J., & Lund, L. J. (eds) (1992) *Proceedings of the International Workshop on Indirect Methods for Estimating the Hydraulic Properties of Unsaturated Soils* (Riverside, California, October 11–13, 1989. University of California).