# Scale and resolution effects of topographic index by 2-D continuous wavelet transform

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**Abstract** Topographic index (TI) is an important parameter in catchment hydrology. The resolution dependence of TI statistics such as mean and variance has been discussed extensively in literature, while the TI structure at different scales, a different issue from resolution, needs more attention. To deal with the directional characteristics of TI structure, this paper uses the two-dimensional continuous wavelet transform (CWT) to explore the TI structure at different scales under several resolutions. For the 90-m resolution, the TI structure can be retrieved up to the scale of 270 m; only the main features (such as main streams) can be retrieved with the scale of 360 m; and even the main structure cannot be retrieved with the 900 m scale. For the 180 m resolution, both high and low order streams can be retrieved at the 180 scale; but for 270 m resolution, only the main steam structure can be captured at the 270 scale. Such preliminary results can give some insight into a suitable DEM resolution for deriving TI and the scale characteristics of TI structure. For example, the 180 m resolution of TI can be used if the high resolution DEM, such as 30 m resolution, is not available; the 270 m resolution TI is not suitable for capturing the flow process in the second or higher order stream.

**Key words** topographic index; scale; direction; 2 dimensional continuous wavelet transform; Morlet wavelet

## INTRODUCTION

Topographic index (TI) is an important parameter in catchment hydrology. For example, TI is a key concept in TOPMODEL (Beven & Kirkby, 1979), a catchment hydrological model that is widely used to predict spatial patterns of soil moisture and depth of the water table (Moore *et al.*, 1991). The prediction of TOPMODEL depends on the mean value and distribution of TI in a catchment. TI is derived from the digital elevation model (DEM); and then the TI statistics depend on the resolution of DEMs (Zhang & Montgomery, 1994). A DEM with low resolution may not capture the small scale features; whereas DEMs with high resolution may not be available.

The scale issue is closely related to the resolution issue, but they are two different concepts. With the decline of resolution (e.g. from 30 m to 90 m), small features, i.e. features at small scales, may disappear. Given a certain resolution, different physical phenomena or characteristics dominate at different scales. It is then worthwhile to explore the TI structure at different scales for a given resolution of TI. Cai & Wang (2006) explored the spatial autocorrelation of TI at different scales under the 30 m DEM; they found that positive spatial autocorrelation of TI exists over a wide range of spatial extents and the autocorrelation declines with the increase of spatial extents, but becomes relatively static beyond a threshold area of approx. 1 km$^2$. To further study the issue of resolution *vs* scale, this paper applies the two dimensional (2-D) continuous wavelet transform (CWT) to exploring the TI structure at different scales and resolutions. The wavelet transform technique is a useful tool to investigate the scale issue. By using the CWT, we test the following hypothesis – for a given scale, a certain DEM resolution may be necessary to derive the TI structure at the scale. On the other hand, for the same purpose, a fine DEM resolution may not be necessary because of data redundancy.

## METHODOLOGY

The wavelet transform originated from seismic data analysis (Morlet, 1981, 1983). Grossmann *et al.* (1985) developed the geometrical formalism of the continuous wavelet transform based on invariance under the translation and dilation, which allowed the decomposition of a signal into contributions of both space and scale. The continuous wavelet transform is particularly suitable for analysing the local differentiability of a function and for detecting and characterizing possible singularities (Arnéodo *et al.*, 1988). The wavelet transform performs like a microscope that discriminates different scales in an *n*-dimensional field, and like a polarizer that separates the different angular contributions of the signal. The wavelet transform of a function $f(b)$ is defined as:

$$Wf(a,b) = \int_{-\infty}^{\infty} f(u)\overline{\psi}_{a,b}(u)du \quad a > 0 \tag{1}$$

where

$$\psi_{a,b}(u) = \frac{1}{\sqrt{a}}\psi(\frac{u-b}{a}) \tag{2}$$

$\Psi_{a,b}(u)$ represents a family of functions, which is called wavelets; $\overline{\psi}_{a,b}(u)$ is the complex conjugate of $\Psi_{a,b}(u)$; and $a$ is a scale parameter. If $a > 1$, it has a dilating effect on the wavelet, otherwise it has a contracting effect. $b$ is a location parameter, and the analysis can be taken around different locations by changing the value of $b$. The wavelet transform defined by equation (1) is called the continuous wavelet transform (CWT) because the scale and location parameters have continuous values. Wavelet transforms implemented on discrete values are called discrete wavelet transforms (DWTs). DWTs have been used to segregate spatial rainfall into multi-scale local averages and multi-scale local fluctuations by Kumar & Foufoula-Georgiou (1993a,b); and technical details of DWTs are presented by Daubechies (1992). A comprehensive review of wavelet applications in geophysical studies can be found in Kumar & Foufoula-Georgiou (1997).

The 2-D CWT is used in this paper. For a 2-D image (such as a TI map) denoted as $f$, the wavelet transform corresponding to the analysing wavelet $\psi$ is defined as (Antoine *et al.*, 1993):

$$Wf(a,\theta,\mathbf{b}) = a^{-1}\int d^2\mathbf{x}\overline{\psi(a^{-1}r_{-\theta}(\mathbf{x}-\mathbf{b}))}f(\mathbf{x}) = a\int d^2\mathbf{k}e^{i\mathbf{b}\cdot\mathbf{k}}\overline{\hat{\psi}(ar_{-\theta}\mathbf{k})}\hat{f}(\mathbf{k}) \tag{3}$$

where $\hat{f}(\mathbf{k})$ denotes a Fourier transform of $f(\mathbf{k})$, and the over bar is the conjugate complex. $Wf(a,\theta,\mathbf{b})$ is the projection of $f$ on the wavelet $\Psi$, which is translated by $\mathbf{b}$, dilated by $a$ and rotated by an angle $\theta$. The 2-D CWT acts as a filter for four variables: location parameters $\mathbf{b} = (b_1,b_2)$, scale parameter $a$ and direction parameter $\theta$. Thus, there are two types of representation: (1) By fixing $(a,\theta)$, the transform is conducted as a function of $\mathbf{b}$ alone. This representation is useful in the detection of position, shape and contours of objects, form recognition and the identification of directional features. (2) With fixing $\mathbf{b}$, the CWT is a function of $(a,\theta)$, i.e. the scale-angle representation. In this representation, by fixing $a$ and traversing along $\theta$, we can extract directional information at a fixed scale $a$; while by fixing $\theta$, scale information in a particular direction can be obtained (Kumar, 1995).

Among several available 2-D analysing wavelets such as the 2-D Mexican hat, difference wavelets and the optical wavelet (Antoine & Murenzi, 1996), the 2-D Morlet wavelet is chosen for this analysis because of its directional selectiveness capability of detecting oriented features and fine tuning to specific frequencies (Antoine *et al.*, 1993). The 2-D Morlet wavelet is defined as:

$$\psi(\mathbf{x}) = \exp(i\mathbf{k}_0\cdot\mathbf{x})\exp(-\frac{1}{2}|A\mathbf{x}|^2) \tag{4}$$

where $i = \sqrt{-1}$, $A = \mathrm{diag}\,[\varepsilon^{-1/2},1]$, $\varepsilon \geq 1$, is a $2 \times 2$ diagonal matrix that defines the anisotropy of the wavelet. $\mathrm{k}_0 = (0,k_0)$ defines the frequency complex exponential $\exp(i\mathbf{k}_0\,\mathbf{x})$. The Morlet wavelet is characterized by the two parameters, $\varepsilon$ and $\mathbf{k}_0$.

In this paper, the TI structure is extracted at different scales. The TI structure, which reflects the stream network, is multidirectional. Thus, for each considered scale, the responses with maximum modulus over all possible orientations are extracted, i.e.:

$$M(\alpha,\mathbf{b}) = \max_{\theta}(|Wf(\alpha,\theta,\mathbf{b})|) \tag{5}$$

The Morlet wavelet transform is computed for $\theta$, spanning from 0 up to $\pi$ at an interval of $\pi/36$ (i.e. 5 degrees); and the maximum coefficients are taken. Because of the periodicity characteristic of Fourier transform, $|Wf(\alpha,\theta,\mathbf{b})|$ is equivalent to $|Wf(\alpha,\theta+\pi,\mathbf{b})|$ (Antoine & Murenzi, 1996).

To demonstrate the directional selectivity of Morlet wavelet, a 2-D random field is created and a horizontal line is added to the field (Fig. 1). $\theta$ indicates the direction in the Fourier domain. Thus large values along this direction in the Fourier domain correspond to large values in ($\pi/2 - \theta$) direction in the physical domain. Figure 2(a) shows the CWT modulus with $\theta = \pi/2$, the horizontal features can be retrieved, i.e. the line in Fig. 1. The orientation selectivity of Morlet wavelet depends on the two parameters: $\varepsilon$ and $\mathbf{k_0}$, specifically the value of $\mathbf{k}_0\sqrt{\varepsilon}$ (Antoine *et al.*, 1993). Comparing Fig. 2(b) to Fig. 2(a), the values of $\varepsilon$ and $\mathbf{k_0}$ are different and others are the same. Figure 2(b) has stronger selectivity in the horizontal direction.

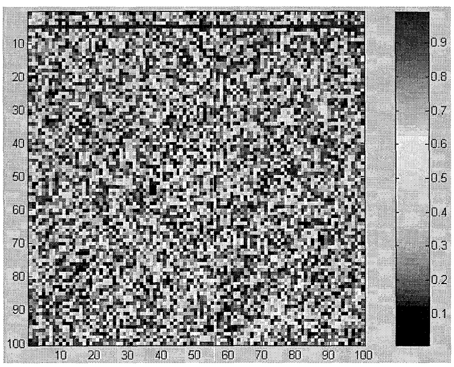


**Fig. 1** 2-D random field with a horizontal line.

(a) (b)

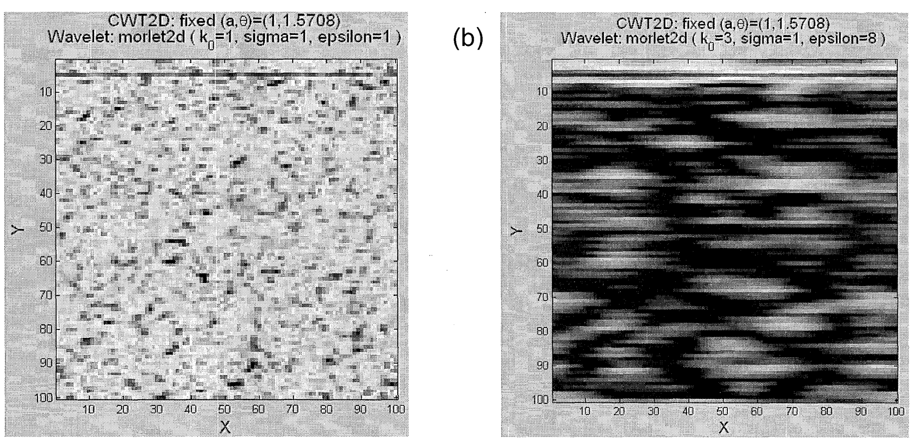


**Fig. 2** CWT with the 2-D Morlet wavelet, (a) $a = 1$, $\theta = \pi/2$, $k_0 = 3$, $\varepsilon = 8$; (b) $a = 1$, $\theta = \pi/2$, $k_0 = 1$, $\varepsilon = 1$.

## RESULT ANALYSIS

A TI map is analysed using the 2-D Morlet wavelet, and the analysis is implemented in ArcGIS and MatLab softwares. Figure 3(a) shows the TI map of a catchment, which includes the Mackinaw River basin located in Illinois. The original resolution of the DEM is about 90 m. Figure 3(b) shows the stream network derived from the DEM when the threshold for defining streams is set to 500 cells (i.e. 4.05 km$^2$ drainage area) using single flow direction algorithm. As shown in Fig. 3(a), the spatial distribution of TI matches the river network. Along the stream (the white lines) the TI values are large; the TI values gradually decrease away from the stream. The black line features are the ridges, which have small TI values. Thus, the directional characteristics are shown with the TI structure.

To extract the TI structure, the optimal values of parameters $k_0$ and $\varepsilon$ are set to 3 and 2, respectively, by trial-and-error. Figure 4 shows the extracted features under a series of scale factors, $\alpha = 1, 2, 3, 4, 10$, which correspond to the scale of 90 m, 180 m, 270 m, 360 m and 900 m, respectively. From small to large scale, the retrieved features become larger and clearer, but the features at small scales cannot be retrieved (e.g. lower order streams). Under the scale of 360 m, only the main features (e.g. main streams) can be retrieved (Fig. 4(d)), but under the 900 m scale, the main structure cannot be retrieved, and the CWT results basically do not reflect the TI structure (Fig. 4(e)).

For the TI map at different resolutions, the $M(\alpha,\mathbf{b})$ can be retrieved at the same scale factor $\alpha = 1$. Figure 4(a) shows the retrieved features for TI at 90 m resolution, and Fig. 5(a) and (b) show the retrieved features at 180 m and 270 m resolution under a corresponding scale (i.e. 180 m or 270 m), respectively. As shown in the figures, under the 180 m resolution, both high and low order streams, but under 270 m resolution only the main steam structure can be captured at the corresponding scales. Comparing Fig. 5(a) to Fig. 4(b), the TI structures are retrieved under the 180 m scale based on TI derived from 180 m and 90 m DEM, respectively. The results from the

(a)

(b)

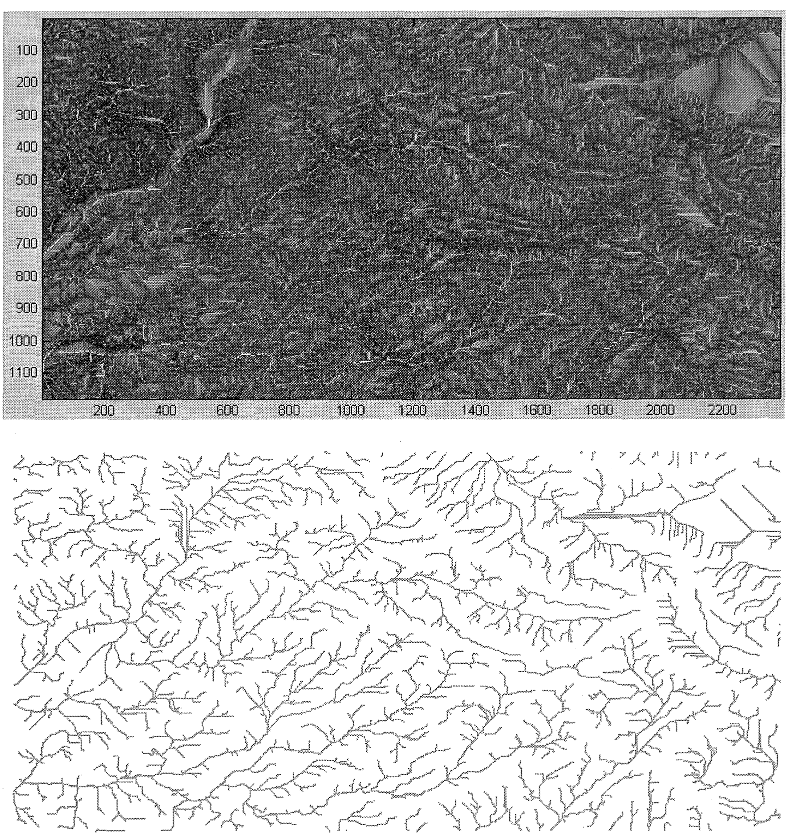


**Fig. 3** (a) A TI map derived from the 90 m DEM. (b) Derived stream network by single flow direction algorithm with a threshold of 500 cells.

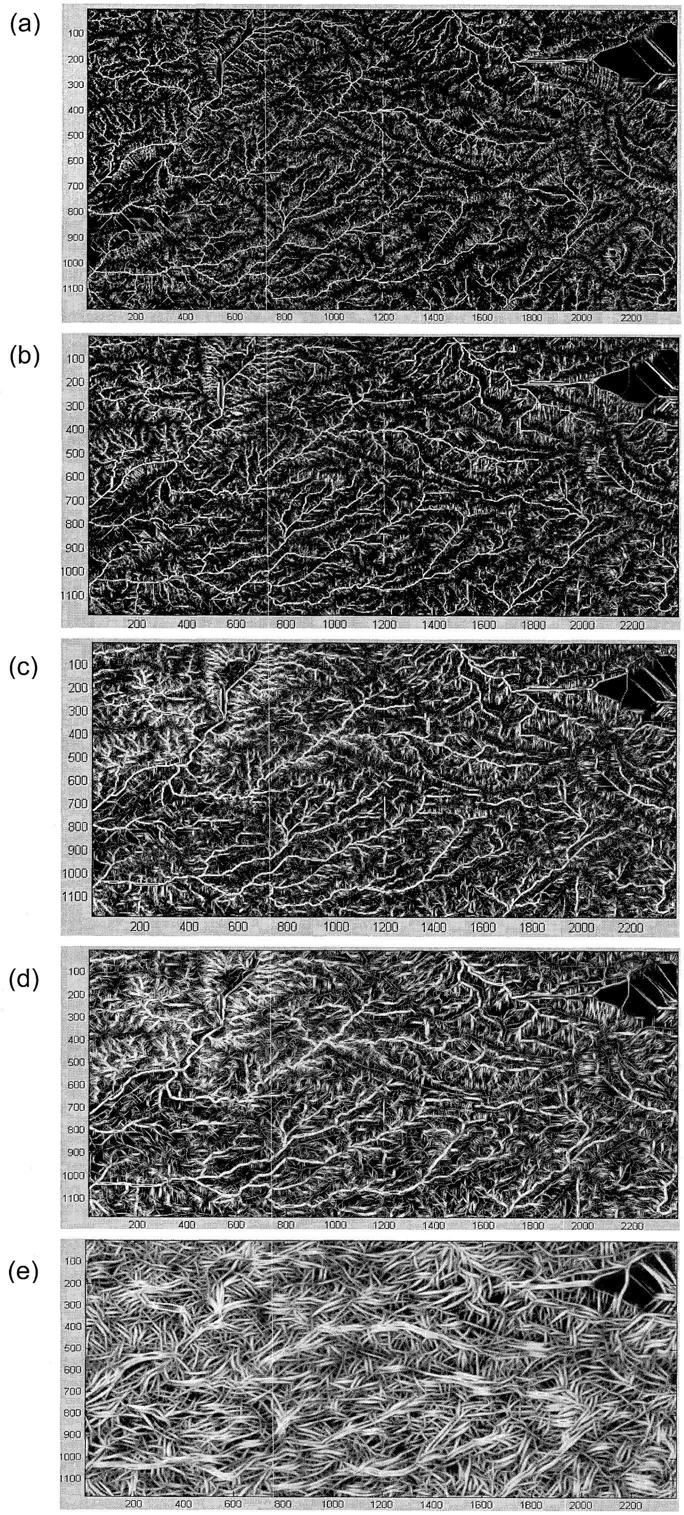


**Fig. 4** TI map under 90 m resolution at the scale (a) $\alpha = 1$; (b) $\alpha = 2$; (c) $\alpha = 3$; (d) $\alpha = 4$; (e) $\alpha = 10$.

180 m DEM are comparable with that from the 90 m DEM. Correspondingly, the features are retrieved at the 270 m scale based on the TI with 270 m and 90 m resolution by comparing Figs 5(b) and 4(c). As can be seen, at the 270 m scale, the TI structure from the 90 m DEM is better than that from the 270 m DEM, especially for the lower order of streams.

(a)

(b)

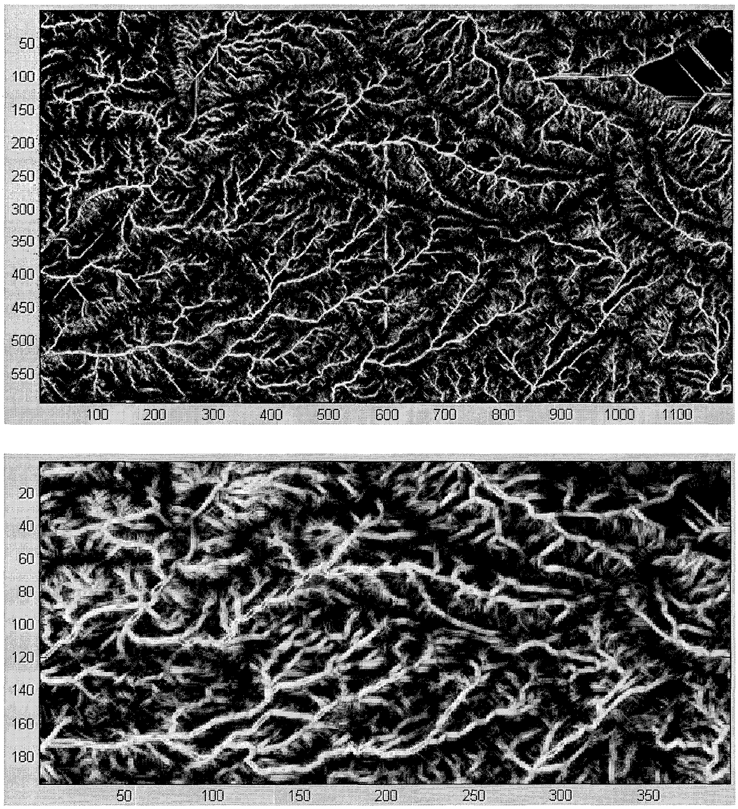


**Fig. 5** TI maps under (a) 180 m resolution and (b) 270 m resolution; $\alpha = 1$ for both cases.

## CONCLUSION

In this paper, the 2-D CWT is applied to investigate the resolution and scale issue using TI as the variable. For the DEM with 90 m resolution, the TI structure can be retrieved up to the scale of 270 m; only the main features (such as main streams) can be retrieved under the scale of 360 m; but the main structure cannot be retrieved under the 900 m scale. For the 180 m DEM, both high and low order streams can be retrieved at the 180 m scale; but for 270 m DEM only the main stream structure can be captured at the 270 m scale. Such preliminary results can provide some insights about a suitable DEM resolution for deriving TI and the scale characteristics of TI structure. For example, for the case study catchment, the TI derived from the 180 m DEM can be used if the high resolution DEM, such as 30 m resolution, is not available; the 270 m DEM is not suitable for capturing the features in the second or higher order streams.

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