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On the development of drainage networks

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ABSTRACT A two-dimensional mathematical model of water and sediment is implemented to describe both flow and soil movement over a hillslope of a given geometry resulting from an effective rainfall pattern. A concentration of flow is induced at the outlet with resulting erosion which propagates upslope. The model permits study of the patterns of growth of the drainage network. When the hillslope contains a random topographic field of elevations or a random field of critical shear stresses in soil properties (or both) the resulting growth upslope quickly develops into a branching structure which fulfils Horton's laws.

INTRODUCTION

Drainage networks are self-forming through the basic mechanisms of sediment transport and erosion. The initiation of erosion is directly related to the bottom shear stress produced by the overland flow; there is a critical stress above which erosion starts on the hillslopes and that critical value depends on the rainfall characteristics, the topography and the soil properties. The cutting of arroyos and their growth upslope are also related to the above phenomena.

The network embodies a deep sense of symmetry, not the trivial symmetry of size, but the much deeper symmetry in formal relations between the parts. This deeper symmetry is contained in the Horton relations but we have yet very little knowledge of the circuits of reciprocal control which must surely exist between climate and the drainage network and which command growth and differentation.

Shreve's (1966) fundamental contribution was to translate into a mathematical framework the concept that Horton's laws of drainage composition are a result of randomness in network topology. His results point to the fact that Horton's laws are largely a consequence of random development of channel networks according to the laws of chance. Impressive as it is, the contribution of Shreve does not contain an explanation since there is no mention of a cause-and-effect relationship. Thus, Shreve's probabilistic framework does not preclude some types of causal mechanisms behind this structural symmetry.

This paper is the first step in a research project which attempts to describe the genetic relationships involved in drainage network growth and differentiation. In other words, the role of chance and necessity in the basic themes which nature preserves through Horton's laws and from which arise the infinite variety of drainage networks.

A two-dimensional mathematical model of water and sediment is implemented to describe both flow and soil movement on a hillslope of a given shape and topography resulting from a certain type of storm (no infiltration is allowed). A concentration of flow is induced at an outlet with resulting erosion which starts migrating upslope. The evolution of overland flow patterns is affected by the growth of the initial channel and the model allows the study through time of the pattern of growth and differentiation of the drainage network.

The mechanism of chance is introduced through either a random topographic field of elevations around a mean sloping plane or through a random field of critical shear stress in soil properties (or both). The fluctuations may be of random origin but their result is no longer purely random. Chance and necessity appear here as complementary principles, that is to say, as integral aspects of the same process.

The study of Horton's laws of stream numbers, stream lengths and stream areas may then be conducted under a cause-and-effect framework with a systematic search for the relative influence of the different variables and mechanisms.

FORMULATION OF THE TWO-DIMENSIONAL MODEL

The mathematical description of overland flow in a basin is best accomplished in a three-dimensional spatial framework. However, the complexities involved in the solution of the three-dimensional models limits considerably its practical use. As alternative formulations, one-dimensional and two-dimensional models have been developed. The principal limitation of the one-dimensional models is the idealization of the physical phenomenon in one space dimension, determining only a general trend in non-uniform channels. There are two general types of two-dimensional models: the vertical view and the plan view. In the latter the flow characteristics are averaged in the vertical direction while in the former the flow characteristics are averaged in one of the horizontal directions.

To study drainage network development and especially branching phenomena, it is necessary to analyse the bidimensional plan view problem. In this way, it is possible to consider the interaction that exists between the increment of the contributing area and the hollow formation (due to the erosive process) at a given point in space. This interaction represents one of the most important effects of overland flow on the dynamics of network growth.

Governing equations

The physical laws that govern overland flow in a basin area are the mass and momentum conservation laws. To study the development of the drainage network, a hydrodynamic model and a sediment transport model need to be coupled in order to account for the hydraulics, sediment transport and morphological phenomena. For this reason, the conservation of mass is represented by two equations: the sediment continuity equation and the water continuity equation. Because of the two-dimensional representation along two perpendicular horizontal directions, the momentum has to be conserved in both of these directions.

Sediment continuity equation This equation is derived by applying the law of conservation of solid mass to a control volume defined in a two-dimensional spatial framework.

$$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \gamma_s \frac{\partial z}{\partial t} = 0 \qquad (1)$$

where g_x is the unit-width solid material transport rate in a x-direction; g_y is the unit-width solid material transport rate in a y-direction; x and y are horizontal space variables; t is time variable; z is ground level elevation referenced to an arbitrary datum; and γ_s is the unit weight of the sediment which can be expressed as

$$\gamma_s = (1 - \lambda)\ \rho_s\ g$$

where g is the acceleration of gravity, λ is the soil porosity and ρ_s is the unit mass of sediment (density).

Water continuity equation This equation is derived by applying the law of conservation of liquid mass to a control volume defined in a two-dimensional spatial framework adding the effect of the rainfall over the free surface.

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial d}{\partial t} = I \qquad (2)$$

where q_x is the unit-width water discharge in the x-direction; q_y is the unit-width water discharge in the y-direction; d is depth of flow; I is effective rainfall intensity.

Momentum conservation equations The momentum conservation equations for a two-dimensional horizontal flow are obtained by vertical integration of the three-dimensional equation of motion (Reynold's equation). Neglecting the Coriolis effect, the surface stresses due to the wind, the rainfall splash effect and the effective shear stresses in vertical planes, the resulting momentum equation in the x-direction is (for more detail see Simons *et al.*, 1979):

$$\frac{\partial q_x}{\partial t} + \frac{\partial (q_x^2/d)}{\partial x} + \frac{\partial (q_x q_y)/d}{\partial y} + gd\,\frac{\partial d}{\partial x} + gd\,\frac{\partial z}{\partial x} + f\,\frac{q_x q_w}{d^2} = 0 \qquad (3)$$

and an identical equation is valid in the y-direction.

The terms in equation (3) represent from left to right: the local inertia, the convective inertia, the cross convective inertia, the pressure, the gravity and the bottom friction. In this equation the only term which is necessary to define is f, which represents a dimensionless friction factor. Following the Chezy friction law, this factor can be expressed as

$$f = g/c^2$$

where c is the Chezy discharge coefficient.

Sediment transport function This equation is used to determine the sediment transport capacity of a specific flow condition. The transport rate includes the suspended load transport rate and the bedload transport rate.

Einstein and Meyer-Peter-Müller equations are the common expressions used to determine suspended and bedload transport rates. An extensive discussion and application on these two equations is provided by Li (1979). Thus, the total load of sediment per unit width of channel can be expressed as:

$$g_* = q_b + q_s \tag{4}$$

where g_* is the total sediment transport capacity, q_b the bedload transport rate and q_s the suspended load transport rate.

The sediment supply rate is another determining factor in the actual sediment transport rate, since to determine its actual rate it is necessary to compare the potential transport capacity of the water (g_*) and the availability of sediment to be transported. Simons *et al.* (1975) found that the sediment supply depends on the initial depth of loose soil left from the previous storm, the amount of soil detachment by raindrop impact and the amount of soil detachment by flow. Li (1979) gives an extensive analysis of the equations to determine sediment supply and the procedure to determine the actual sediment transport rate.

From Einstein, Meyer-Peter-Müller and equation (5) it is possible to conclude that from a given soil, slope and fluid the total sediment transport capacity will be a function of the velocity and depth of fluid. This argument will be used later on, to reduce the complexity of the sediment transport equations through simplifying assumptions.

Formulation of the two-dimensional model

The solution of the two-dimensional model can be obtained from equations (1), (2), (3) and (4). These equations will yield a set of four partial nonlinear differential equations with four unknowns: the depth of the water, the unit width of water discharge in x and y-directions and the depth of sediment.

The numerical solution of the above set of equations is a difficult task that involves a great deal of computational time with serious problems of instabilities in the solution scheme. Thus, to pursue the integration of these equations some assumptions are necessary to decrease the complexities of the equations yet maintaining a physically realistic representation of the phenomenon.

Kinematic assumption of the hydrodynamic equations In the upper part of a fluvial system, or what is called the production zone of water and sediment (Schumm, 1977), the slope is generally sufficiently high that the controlling forces in the momentum equation are the gravity and friction forces. Thus, the main assumption of the kinematic approximation is that the friction slope is equal to the channel bed slope, meaning that the pressure and inertia (local and convective) term are neglected.

The simplified hydrodynamic equations are now:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial d}{\partial t} = I \tag{5}$$

$$gd\,\frac{\partial z}{\partial x} + f\,\frac{q_x q_w}{d^2} = 0 \tag{6}$$

$$gd\,\frac{\partial z}{\partial y} + f\,\frac{q_y q_w}{d^2} = 0 \tag{7}$$

defining S_x and S_y as

$$S_x = -\frac{\partial z}{\partial x} \text{ and } S_y = -\frac{\partial z}{\partial y}$$

and combining equations (5), (6) and (7) yields

$$\frac{\partial}{\partial x}\left[-\sqrt{(g/f)}\frac{S_x\, d^{1.5}}{(S_x^2 + S_y^2)^{0.25}}\right] + \frac{\partial}{\partial y}\left[-\sqrt{(g/f)}\frac{S_y\, d^{1.5}}{(S_x^2 + S_y^2)^{0.25}}\right] + \frac{\partial d}{\partial t} = I \tag{8}$$

All factors of equation (8) are constant at each point except d (depth of fluid), and it is possible to express this equation as:

$$\frac{\partial d}{\partial t} = A\,\frac{\partial d}{\partial x} + B\,\frac{\partial d}{\partial y} + I \tag{9}$$

where

$$A = \frac{1.5\ S_x}{(S_x^2 + S_y^2)^{0.25}}\ \sqrt{(gd/f)} \tag{10}$$

$$B = \frac{1.5\ S_y}{(S_x^2 + S_y^2)^{0.25}}\ \sqrt{(gd/f)} \tag{11}$$

Sediment transport equation approximation The main objective of this investigation is not to predict the volume of sediment transported in a watershed but the study of the growth of the drainage network. Thus what is needed is the accounting of relative magnitude of the erosion process in time and space. For this reason, a simple model based on an empirical power function of the mean velocity is preferred to the more sophisticated phenomenological equations. Thus, the selected transport function is taken as

$$g_w = c_g\ w^m \tag{12}$$

where g_w is the sediment transport rate per unit width in the direction of the velocity w, c_g and m are empirical coefficients.

Solving equation (12) along two perpendicular directions and recalling that q = w d, yields

$$g_x = c_g\ \frac{q_x\ q_w^{m-1}}{d^m} \tag{13}$$

$$g_y = c_g\ \frac{q_y\ q_w^{m-1}}{d^m} \tag{14}$$

where

$$q_w = (q_x^2 + q_y^2)^{\frac{1}{2}} \tag{15}$$

NUMERICAL MODEL

Modelling approach

Several authors (Chen, 1973; Simons *et al.*, 1978, 1979) have shown for one-dimensional and bidimensional models, that the hydrodynamics and sediment transport equations can be solved in a sequential manner. This approach simplifies the procedure of solution with results as good as those obtained by solving both equations simultaneously.

The solution procedure consists of two stages:

(a) fixing the bed-level, the hydrodynamic equations are solved updating the values of d, q_x and q_y;

(b) fixing d, q_x and q_y, the sediment transport and sediment continuity equations are solved updating the values of the bed levels (z).

Solution schemes

The numerical solution of the hydrodynamic equation can be accomplished by using a finite difference scheme of the explicit or implicit type. The problem with the explicit schemes is the strict stability criterion which limits the size of the time step, whereas the implicit scheme requires an iterative solution of a large set of simultaneous equations involving serious problems of convergence.

An alternative scheme is the ADI method (alternating direction implicit) which is a semi-implicit scheme. This method decomposes a two-dimensional problem in two sequential one-dimensional problems. The main characteristics of the finite difference grid used in this study are indicated in Fig.1. In each node of this grid are defined the initial values of z(x,y,o), d(x,y,o), q_x(x,y,o) and q_y(x,y,o).

The ADI equations for computing the interior points of the grid are the following:

$$\left[\underline{I} - \frac{\Delta t}{2} A_{ij}^{k+\frac{1}{2}} D_{ox}\right] d_{ij}^{k+\frac{1}{2}} = \left[\underline{I} + \frac{\Delta t}{2} B_{ij}^{k} D_{oy}\right] d_{ij}^{k} + \frac{\Delta t}{2} I \tag{16}$$

$$\left[\underline{I} - \frac{\Delta t}{2} B_{ij}^{k+1} D_{oy}\right] d_{ij}^{k+1} = \left[\underline{I} + \frac{\Delta t}{2} A_{ij}^{k+\frac{1}{2}} D_{ox}\right] d_{ij}^{k+\frac{1}{2}} + \frac{\Delta t}{2} I \tag{17}$$

where $\underline{I}$ is an identity matrix, Δt is a time step, i is the index in the x-direction, j the index in the y-direction, k the time index, A and B were defined in equations (10) and (11) and D_{ox} (or D_{oy}) is an operator defined by:

$$D_{ox} d_{ij} = \frac{d_{i+1,j} - d_{i-1,j}}{2\Delta x} \tag{18}$$

The numerical solution of the sediment continuity equation was accomplished by using an explicit finite difference shceme, where the discretization was performed in the following way:

$$\left[\frac{\partial g_x}{\partial_x}\right]_{ij} = D_{ox} g_{x_{ij}} \tag{19}$$

$$\left[\frac{\partial g_y}{\partial y}\right]_{ij} = D_{oy} g_{y_{ij}} \tag{20}$$

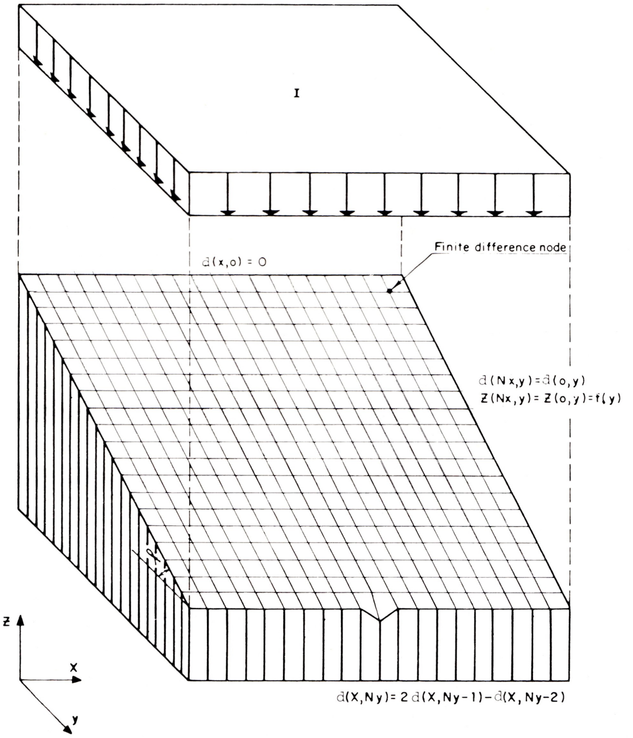

FIG.1 Schematic of drainage basin with uniform rainfall input and an artificially induced outlet.

Thus the final equation that gives the ground level elevation z_{ij}^{k+1} as a function of z_{ij}^{k} is:

$$z_{ij}^{k+1} = z_{ij}^{k} - \frac{\Delta t \ (D_{ox} g_{x\ ij} + D_{oy} g_{y\ ij})}{(1 - \lambda)\ \rho_s g} \tag{21}$$

Boundary conditions

The example developed in this paper is a rectangular basin with constant slope in the y-direction, zero slope in the x-direction and having effective rainfall as the only source of water supply (Fig.1).

Thus, the boundaries of the grid (o,y) and $(N\Delta x,y)$, (N+1 is a number of nodes in x-direction) will be open, but with fixed values of z, and at the boundary (x,o), d = O, and at $(x,N\Delta y)$ the boundary is open with variables z's. The boundary conditions are then:

$$d(x,o,t) = o$$
$$d(o,y,t) = d(N\Delta x,y,t)$$
$$N + 1 = \text{number of nodes in the x-direction}$$
$$q_x(o,y,t) = q_x(N\Delta x,y,t) = o$$
$$z(o,y,t) = z(N\Delta x,y,t) = f(y) \ \text{(specified)}$$
$$d(x,M\Delta y,t) = 2\ d(x,(M-1)\ \Delta y,t) - D(x,(M-2)\ \Delta y,t)$$
$$M + 1 = \text{number of nodes in the y-direction}$$

and the initial conditions are:

$z(x,y) = f(y)$ (zero slope in x-direction)
$d(x,y) = g(y)$

The proposed scheme for the sediment continuity equation is just the solution of

$$\frac{\partial z}{\partial t} = K \tag{22}$$

where

$$K = -\frac{\left[(\partial g_x/\partial x) + (\partial g_y/\partial y)\right]}{(1-\lambda)\,\rho_s g} \tag{23}$$

taking $(\partial g_x/\partial x)$ and $(\partial g_y/\partial y)$ as constant during the time interval Δt as given by expressions (19) and (20). This assumption can be justified because the hydrodynamic condition varies very little during the time interval in which z changes from z^k to z^{k+1}.

CONTROLLED EXPERIMENTS

Controlled experiments were carried out modelling mathematically the surface runoff and sediment transport processes in the rectangular watershed described above.

In the first experiment the soil characteristics were considered constants over the whole area, having a constant initial slope in the y-direction and a constant rainfall intensity (in time and space). In order to induce an initiation of the network growth, an artificial outlet was produced in the middle end of the watershed $(N\Delta x/2,\ N\Delta y)$ by lowering the values of z.

Fig.2 shows the results obtained in this experiment, after 120

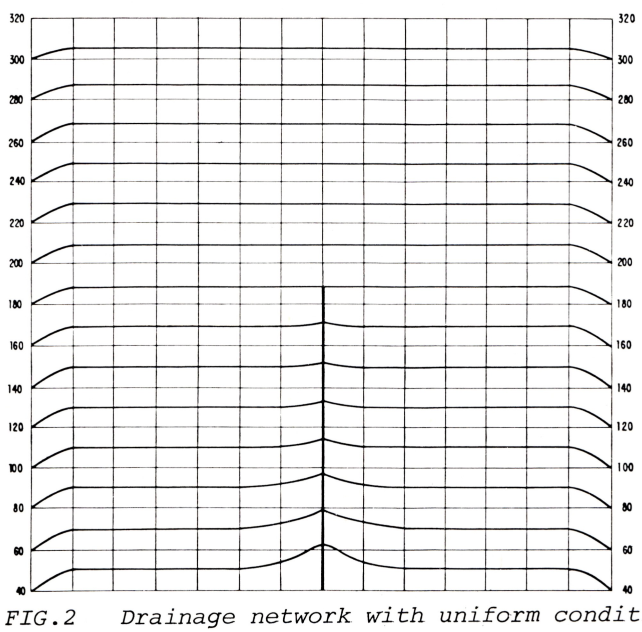

FIG.2 Drainage network with uniform conditions of topography and soil characteristics.

time steps. This illustrates the process of channel formation, where the areas adjacent to the hollow are changing in time (because of the erosion process) with the corresponding increase of the contributing area to the basin outlet.

An important element of the growth process is the existing feedback mechanism between surface runoff process and network development. The flowing water produces erosion and sediment transport in the main slope direction. This erosion changes the shape of the contour lines increasing the contributing area to the hollow and producing an increment in the flowing water. Thus, the process can be considered as regenerative and self-activated.

The results of the first experiment (Fig.2) indicate that for totally homogeneous soil and topography only one channel is formed, beginning at the site where the artificial outlet was provoked and growing up to the point where the shear stress of the water is not large enough to produce erosion (upper part of the basin). In the light of this result one question arises: how does branching arise in the drainage network? The answer could lie in the structured randomness of the climate-soil-vegetation system and the boundary conditions that define the shape of the basin. The time and spatial variability of the effective rain intensity, texture and structure of

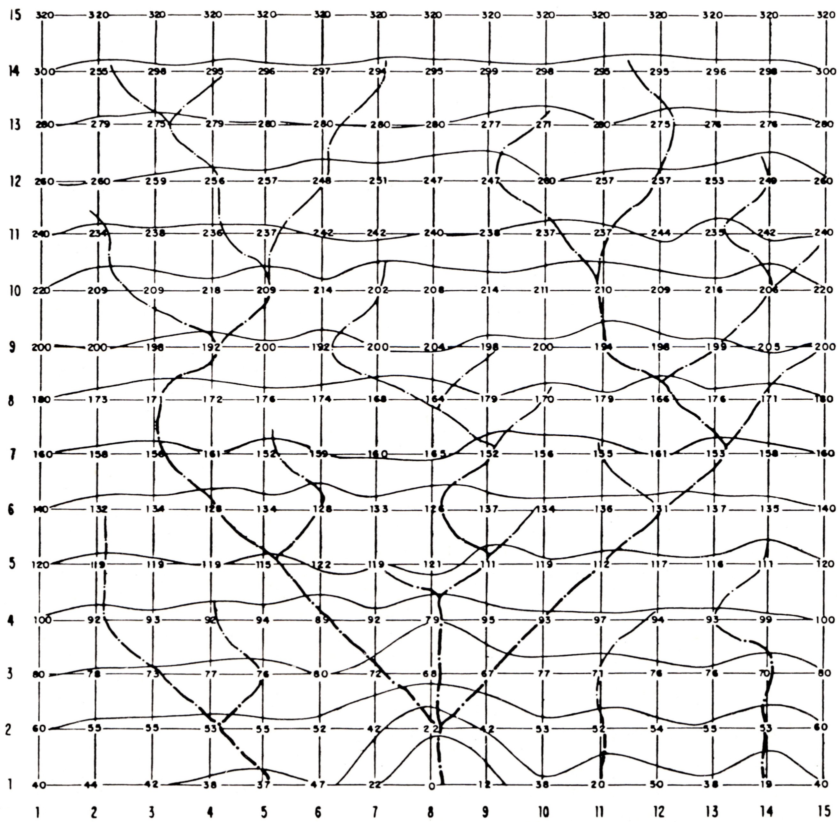

FIG.3 Drainage network with stochastic soil characteristics.

the soil, vegetation cover and topography interact producing a differential effect on the erosion power of the water.

In the model described above, the effect of topography (slope) is explicitly considered, whereas all other factors involving soil and vegetation are implicitly included in the empirical coefficient of the sediment transport equation (c_g in equation (12)). In effect, c_g depends on the friction factor, which accounts for the type of soil and vegetation cover.

The second experiment that was carried out included a random description of the parameter c_g. As a first approximation, a uniform description (homogeneous in space) was assumed for the probability description of c_g. Thus a value of c_g was generated for each node of the grid from the uniform distribution.

An example of the results of the second type of experiment is shown in Fig.3. The branching structure appears quite rapidly in the development of the drainage network and although the causal mechanism has a random origin, its results is no longer purely random. The network embodies a deep sense of symmetry which is expressed in the structural relations between its parts. Thus, the networks generated in the second type of experiment fulfil Horton's laws of stream number and stream lengths. Table 1 gives Horton's analysis for the network obtained in Fig.3.

TABLE 1 Horton's analysis for the network obtained in Fig.3

w	N_w	L_w
1	16	2.50
2	4	6.25
3	1	10.00
	$R_B = 4$	$R_L = 2.05$

Research now in progress is addressed towards the mathematical description of the coupled relations between the parameters of the water-sediment system which control growth and differentiation.

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