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A hydrodynamic model of sediment transport in rill flows

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ABSTRACT This study discusses the basic problem of the transport of sediment in small flow channels which naturally exist on intensively cropped land. Emphasis is placed on the transport mode in the form of a densely packed moving layer near the bottom of the channel. Governing equations for the thickness of the bed layer are derived which are based upon the results of some recent studies pertaining to the constitutive laws for stress and deformation rate of granular materials.

INTRODUCTION

Erosion and transport of soil by water on intensively cropped land has received considerable attention in the past. The main thrust of investigations has been the development of simplified empirical relationships for sediment yield and their experimental verification. These equations have proved to be quite useful for management purposes, but lack the scientific detail essential for a well founded mathematical model derived from physical laws. A systematic approach to erosion and transport of soil from upland areas is important because it represents the complex interaction of the kinetics of falling rain, the hydrodynamics of flow and the dynamics of granular materials.

Essentially all of the sediment lost from intensively cropped land during rainstorms is transported by runoff along row furrows, rills and other flow concentrations. The rate of sediment transport for such conditions depends on the rate at which the sediment is eroded from the interrill areas to the concentrated flows, the size and density of the sediment and hydrodynamic characteristics of the flow channels. Under normal conditions row cross sections cause flow to be parallel to the rows. Where the land slope is not high, the lengths of the rows and their steepnesses are generally small and render the sediment movement and the water flow strongly interdependent. Soils are eroded by raindrop impact from the interrill areas between rills or between rowcrop furrows and from rills. Rates of interrill erosion may be quite high on intensively cropped land (Meyer & Harmon, 1979), but the sediment yield from the row furrows will depend on the hydraulic characteristics as well as the size, density and related bulk properties of the eroded materials.

For example, a large proportion of the finer particles will be transported in suspension even at low flow rates. But the transport of larger and/or denser sediments such as sand or aggregates takes place as bed load. These sediments move only under sufficiently high flow rates in the row furrows. A part of the eroded sediment may be deposited, in which case the sediment yield is much lower. Thus, the flow of sediment in row furrows needs a detailed hydrodynamic analysis, which is the main theme of the present paper.

GOVERNING EQUATIONS

The equations governing flow of a turbulent two-phase mixture of water and granular sands are given by

$$\frac{\partial c}{\partial t} + (c u_k^s)_{,k} = 0 \tag{1}$$

$$\frac{\partial(1-c)}{\partial t} + \left[(1 - c)u_k\right]_{,k} = 0 \tag{2}$$

$$c\rho^s\left[(\partial u_k^s/\partial t) + u_\ell^s u_{\ell,k}^s\right] = -cp_{,k} + S(u_k - u_k^s) - c\rho^s g f_k + \rho^s \tau_{k\ell,\ell}^s \tag{3}$$

$$(1 - c)\rho\left[(\partial u_k/\partial t) + u_\ell u_{\ell,k}\right] = -(1 - c)p_{,k} + S_d(u_k^s - u_k) - (1 - c)g f_k + \rho\tau_{k\ell,\ell} \tag{4}$$

In the above c is the sediment volume fraction, u_k^s and u_k are the sediment and fluid mean velocities, ρ^s and ρ are the sediment and fluid densities, p is the mean fluid pressure, S_d the drag force per unit slip velocity and $\tau_{k\ell}^s$ and $\tau_{k\ell}$ are the stress tensors acting on the solid and the fluid, respectively. A comma indicates partial differentiation whereas a repeated index stands for summation. (1) and (2) indicate the balance of mass whereas (3) and (4) the balance of momentum of the mixture. (1-4) are obtained in a standard way by applying the principles of continuum mechanics as applied to mixture theory (Soo, 1967).

In the following we wish to apply the above equations to a special case in which the concentration c increases rapidly to a maximum value in a thin layer near the bottom of the channel. It is assumed that the granular solids are transported by the hydrodynamic force of water but form a thin zone with variable thickness δ over the immovable bed. The sediment layer consists primarily of larger size particles, whereas fine particles and clay are transported in suspension. Due to variations in the dynamic conditions of the two modes of transport, there exists a time dependent exchange between the suspended and the bed layer sediments. To study the mechanics of bed layer transport we will, however, assume that there is no dynamic interaction between them. Thus, the total sediment yield will be given by a linear sum of the two. Further, only two dimensional motion in the vertical plane will be considered.

We consider the motion in the x-z plane as shown in Fig.1. The

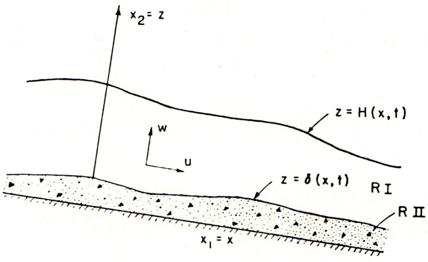

FIG.1 Sediment transport in a small channel; R_I is the water and sediment (dilute suspension) suspension region; R_{II} is the bed layer.

details of motion are given by a composite picture of the three flow fields: (a) flow of water in $0 \leq z \leq H$; (b) flow of suspended sediment in $\delta < z \leq H$; and (c) flow of sediments in bed layer, $0 \leq z < \delta$.

Flow of water

It is assumed that only small size particles are present in suspension and the concentration c is quite small in $\delta < z \leq H$. Therefore, it is anticipated that the effect of slip between the fluid and sediment is quite small. Thus, we assume

$$u_k^s = u_k \tag{5}$$

so that the governing equations for water flow yield

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = q \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = g(S_O - S_f) \tag{7}$$

where

$$h = H - \delta$$

$$Q = \int_\delta^H u dz$$

S_O and S_f are bed and friction slopes respectively. Equations (6)-(7) are the same as the St Venant equations utilized in open channel hydraulics. q is the lateral inflow from the surrounding interrill areas. The effect of momentum due to the lateral inflow in (7) has also been neglected.

Flow of suspended sediment

Since fine sediments including clay particles will be transported in suspension, we assume that $u_k^s = u_k$ and the concentration c is quite small. We integrate (1) with respect to z from $z = \delta$ to $z = H$ and obtain

$$\frac{\partial}{\partial t}\int_{\delta}^{H} c\,dz + \frac{\partial}{\partial x}\int_{\delta}^{H} cu\,dz + \left(c_{\delta}\frac{\partial \delta}{\partial t} - c_{H}\frac{\partial H}{\partial t}\right)$$

$$+ \left(c_{\delta}u_{\delta}\frac{\partial \delta}{\partial t} - c_{H}u_{H}\frac{\partial H}{\partial x}\right) + \left(c_{H}w_{H} - c_{\delta}w_{\delta}\right) = 0 \qquad (10)$$

In (10) c_H and c_δ are the concentrations at the top and bottom surfaces of the suspension domain. The boundary conditions at these two surfaces are given by

$$c_H\left(\frac{\partial H}{\partial t} + u_H\frac{\partial H}{\partial x} - w_H\right) = q_i \qquad (11)$$

$$c_\delta\left(\frac{\partial \delta}{\partial t} + u_\delta\frac{\partial \delta}{\partial x} - w_\delta\right) = q_b \qquad (12)$$

where q_i is the sediment influx due to erosion in the surrounding interrill areas, and q_b is the amount of sediment flux either deposited or picked up by the suspended sediment from the bed layer. When (11)-(12) are utilized, (10) leads to

$$\frac{\partial C}{\partial t} + \frac{\partial Q_s}{\partial x} = q_i - q_o \qquad (13)$$

where

$$C = \int_{\delta}^{H} c\,dz\ , \qquad Q_s = \int_{\delta}^{H} cu\,dz \qquad (14)$$

It may be noted that in the present case the sediment is received by the flow channel from the surrounding areas at the top surface and, therefore, the concentration at z = H is not zero. The product of c_H and the rate of lateral inflow of water yields the lateral influx of sediment given by q_i. The determination of q_b is quite involved, because it represents the rate of exchange of sediment between the bed layer and the surroundings. The vertical flux of sediment away from the bed is related to the sediment parameters through a consideration of the mass and momentum equations for the water and the sediment. Previous workers (Bagnold, 1973; Owen, 1964; Parker, 1975) have examined the problem of the transport of sediment over a flat bottom using calculations of the trajectory of a saltating particle. In this context the concept of turbulent fluctuations in the flows have also been introduced, but there is no simple theory available which may be utilized at present. Therefore, we wish to study in some detail the dynamics of bed layer transport by neglecting the exchange process which exists with the suspended mode of sediment transport. A partial justification for this type of analysis may be advanced in a physical case in which the stream reaches its transport capacity in suspension, so that an additional influx of sediment must be carried by the bed load. In this case C in (13) may be taken to be constant so that (13) yields

$$q_i = q_o \qquad (15)$$

q_o in (15) must, then, be interpreted as the net amount of sediment deposited on the channel bed which is transported in the bed layer. The momentum equation of the suspended sediment does not lead to any new result in this case and yields the same equation as (7).

Flow of sediment in the bed zone

We assume that in this zone, solid grains are propelled by the flow of water over the fixed bed of grains of the same sort. In general there exist three modes of movement: rolling, saltation and suspension. In addition the grains also have angular motions. Since we assume that the bed layer is densely packed, the effect of spin and suspension is assumed to be small and, therefore, the present analysis is restricted to a consideration of an average of the transport modes in saltation and rolling as evidenced by the experiments of Francis (1973). The solid grains are assumed to be packed together and move along the rigid bed in the form of bulk flow due to the hydrodynamic force of the water and the shear stress at the boundary $z = \delta$. In addition the weight of the grain itself may also add to the propelling mechanism.

The motivation for the present analysis, which treats the bed layer as a bulk flow model, has been derived from the work of Bagnold (1954), Goodman & Cowan (1972), and Savage (1979). These authors were concerned with the development of certain constitutive equations appropriate for flow of cohesionless granular materials with finite deformation rates at low stress levels. The theory accounts for the non-Newtonian nature of the flow as evidenced by Bagnold's (1954) experiments. The recent formulations of the constitutive equations have been applied successfully to gravity flows in inclined chutes and vertical channels. These flows, moreover, are with relatively high deformation rates so that the grain inertia plays a dominant role. But sediment flows along channel beds take place at relatively slower rates which permits exclusion of the inertial effect. It seems plausible, therefore, to assume that the concentration c in the bed layer is constant but the non-Newtonian nature of flow must be retained. Thus, when (1) is integrated with respect to z between 0 to δ, we obtain

$$\frac{\partial}{\partial x}\int_0^{\delta} c_b u^s dz - (c_b u^s)_{z=\delta}\frac{\partial \delta}{\partial x} + (c_b w^s)_{z=\delta} = 0 \tag{16}$$

The boundary condition at $z = \delta$ is given by

$$c_b\left(\frac{\partial\delta}{\partial t} + u^s\frac{\partial\delta}{\partial x} - w^s\right) = q_b \tag{17}$$

where c_b is sediment concentration in the bed, and q_b is the sediment influx to the bed layer. When (17) is utilized, (16) yields

$$c_b\frac{\partial\delta}{\partial t} + \frac{\partial}{\partial x}Q_b = q_b \tag{18}$$

$$Q_b = \int_0^{\delta} c_b u^s dz \tag{19}$$

For an approximate evaluation of the integral in (19), we now consider the momentum equations pertaining to the flow of water and the sediment layer. We adopt the point of view that the sediment layer is driven by the hydrodynamic forces. These consist of forces which directly act on the sediment particles plus a certain shear stress acting on the surface $z = \delta$. Under normal conditions the sediment particles near the bed ($z = 0$) will tend to adhere to the surface of the bed. In this case $u^s = 0$ when $z = 0$. But during high flows, certain slip conditions may exist between the bed layer and the bed. In the following analysis we neglect the effect of this slip for the sake of simplicity, but the results are easily extendable for non-zero slip conditions.

Since the classical theory of channel flow neglects the z component of velocity w, we also compute the hydrodynamic pressure p on the sediment particles in a similar way. Thus for water, the z component of momentum reduces to

$$\partial p/\partial z = -\rho g \tag{20}$$

which yields the pressure at $z = \delta$

$$p_\delta = \rho g(H - \delta) \tag{21}$$

Similarly, the pressure variation in the sediment layer is given by

$$p = -\rho^s gz + H\rho g + (\rho^s - \rho)g\delta \qquad 0 \lesssim z \lesssim \delta \tag{22}$$

from which

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial H}{\partial x} + (\rho^s - \rho)g\frac{\partial \delta}{\partial x} \qquad 0 \lesssim z \lesssim \delta \tag{23}$$

The momentum equation along the x-axis yields

$$0 = -c_b\frac{\partial p}{\partial x} + c_b\rho Sg + \rho^s\frac{\partial \tau^s}{\partial y} \qquad 0 \lesssim z \lesssim \delta \tag{24}$$

where S is the channel slope and τ^s is the shear stress on the sediment particles. As discussed previously, in the spirit of open channel hydraulic analysis, we have neglected the inertial effect of sediments in arriving at (24). When (23) is substituted in (24), an integration of the result yields

$$\tau^s = \{c_b\rho g\frac{\partial H}{\partial x} + c_b(\rho^s - \rho)g\frac{\partial \delta}{\partial x}\}y - c_b\rho^s gSy + A \qquad 0 \lesssim z \lesssim \delta \tag{25}$$

The constant A may now be evaluated by a knowledge of the shear stress τ_δ at $z = \delta$. The shear stress τ_δ depends on the type of flow given either by (6)-(7) or from an analysis of boundary layer theory based upon the fluid dynamics of turbulent flows. It seems also plausible to estimate τ_δ from Shield's diagram. Thus,

$$A = -\{c_b\rho g\frac{\partial H}{\partial x} + c_b(\rho^s - \rho)g\frac{\partial \delta}{\partial x} - c_b\rho^s gS\}\delta + \tau_\delta \tag{26}$$

At this stage we must consider the mechanics of the flow of granular solids such that the shear stress τ^s may be related to the deformation rates. Much of the previous work on granular flows has dealt with confined flow in bins and hoppers. Typically, predictions of the stress fields have been obtained by solving the quasi-static equilibrium equations under the assumption that the bulk solid obeys the Coulomb yield or failure criterion in accordance with the Mohr-Coulomb theory of limiting equilibrium. Yielding in influenced by net hydrostatic pressure in the Coulomb failure criterion, which states that yielding takes place when

$$|\tau| = \tau_O + T \tan \phi$$

when τ and T are respectively the shear and the normal stresses, τ_O is the cohesion and ϕ is the internal angle of friction of the bulk solid. While this approach has been applied in many statical problems, uncertainty exists about its extension in dynamical cases, particularly with respect to the choice of ϕ.

Over 25 years ago Bagnold (1954) performed experiments on neutrally buoyant spherical particles suspended in a mixture of glycerine, water and alcohol and sheared in a coaxial rotating cylinder apparatus. Both the torque and the normal stress in the radial direction were measured when various concentrations of the spherical grains were sheared. In the grain inertia region in which the fluid in the interstices plays a minor role and the dominant effects arise from particle-particle interactions, Bagnold observed that the shear and the normal stresses are proportional to the square of the velocity gradient. The interesting phenomenon was the presence of a normal stress (which Bagnold termed the dispersive pressure) proportional to the shear stress, reminiscent of the quasi-static behaviour of a cohesionless Coulomb material. Bagnold's analysis indicated that the normal stress in the z-direction was

$$P = a\rho^s \lambda f(\lambda) D^2 \left(\frac{\partial u}{\partial z}\right)^2 \cos \alpha_i \tag{27}$$

and the grain shear stress

$$\tau^s = \rho \tan \alpha_i \tag{28}$$

where f is an unknown function of λ, a is a constant and α_i is an unknown angle dependent upon the collision conditions. D is the diameter of the grain and λ is certain linear concentration. Bagnold was able to determine appropriate values of a, α_i and $f(\lambda)$. Bagnold applied these results to predict the flow of dry sand down an incline.

Therefore, on the basis of Bagnold's pioneering research, we assumed

$$\frac{\partial u^s}{\partial z} = K(\tau^s)^\alpha \tag{29}$$

where K and α are constants to be determined experimentally. α may be assumed to be equal to 2 in the present case. Equation (29) may be integrated with respect to z to yield

$$u^s = \frac{K}{\alpha + 1} \frac{\{\tau_\delta - c_b g[\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S] \quad \}^{\alpha+1}}{c_b g[\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S]} + A_1 \tag{30}$$

The constant A_1 may be evaluated from a knowledge of the relative slip between the sediment layer and the bed. If we neglect the slip, then we have

$$u^s = 0, \qquad z = 0 \tag{31}$$

which yields

$$A_1 = - \frac{K}{\alpha + 1} \frac{\{\tau_\delta - c_b g[\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S]\, \delta \}^{\alpha+1}}{c_b g[\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S]} \tag{32}$$

The sediment discharge Q_b is finally obtained as

$$Q_b = \int_0^\delta c_b u^s dz$$

which yields

$$Q_b = \frac{K c_b}{(\alpha + 1)(\alpha + 2)} \frac{(\tau_\delta^{\alpha+2} - \{\tau_\delta - c_b g[\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S]\, \delta\}^{\alpha+2}}{c_b^2 g^2 [\rho \frac{\partial H}{\partial x} + (\rho^s - \rho)\frac{\partial \delta}{\partial x} - \rho^s S]^2}$$

$$+ A_1 \delta c_b \tag{33}$$

Note that in the general case the sediment discharge Q_b depends on the bed layer thickness δ as well as $\partial\delta/\partial x$. The dependence on other parameters, i.e. τ_δ, S and H is also there which, of course, was expected.

The formulation of the mathematical problem of determining the sediment discharge may now be considered complete. There are several interesting features of the present formulation which need detailed analysis and discussion. To illustrate a rather simple case, consider the transport of the sediment only due to the action of the shear stress τ_δ so that we assume $\partial\delta/\partial x = \partial H/\partial x = S = 0$ in (33). In this case,

$$\frac{\partial \delta}{\partial t} + b\delta \frac{\partial \delta}{\partial x} = \frac{q_0}{c_b} \tag{34}$$

where

$$b = K\tau_\delta^\alpha \tag{35}$$

(34) is a hyperbolic partial differential equation and its solution may be discussed in some detail with respect to given initial and boundary conditions. A good reference for this purpose is the book by Whitham (1974).

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