

Hydrodynamic averaging of overland flow and soil erosion over rilled hillslopes

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Abstract The rill structure has a profound influence on the flow and erosion pattern over a hillslope, and therefore needs to be represented in physical models in an appropriate manner. Using spatial scale arguments and the ergodic hypothesis, key properties of the rill structure may be identified which allow for the replacing of the rapidly varying microtopography with more stable averages. This paper suggests a new model for quantifying the influence of rill structure in physical models through stochastic averaging of the hydrodynamic processes. Comparisons of the model predictions with experimental observations over a hillslope show good agreement.

INTRODUCTION AND BACKGROUND

Prediction of overland flow and sediment transport from hillslopes has important implications in resolving water quality and land use issues. For example, many freshly cut hillslopes are located adjacent to highways, and the soil erosion from such hillslopes poses a severe traffic hazard on the road at the foot of these slopes. One important approach to analysing such problems is through the use of physical models. However, many of the current techniques for modelling overland flow and sediment transport are inadequate because they do not represent the rill structure over the hillslopes in an appropriate manner. Such models neglect the differences between the transport capacities of the flow within rills and the flow over adjacent overland flow sections, often leading to serious misinterpretation of results. Laboratory and field experimental results have demonstrated that the presence of rills significantly affects erosion of the soil surface (Meyer *et al.*, 1975; Moss & Walker, 1978). Therefore effective abatement strategies can be formulated only after a better understanding of the rill pattern is obtained and when physical models are able to represent the rill geometry.

Towards this end, the rill structure over a particular hillslope was studied and different spatial scales were identified in order to characterize the hillslope microtopography. An ergodic length scale was established so that continuum representation could be used over the hillslope, instead of modelling the detailed soil profile. The influence of microtopography on overland flow and subsequent erosion and sediment transport computations has been recognized

by Emmett (1978), Dunne & Dietrich (1980), Abrahams *et al.* (1989) and Abrahams & Parsons (1990). These authors have criticized the sheet flow approximation of conventional overland flow models.

The present work dealing with surface structure of rilled hillslopes has important implications for the hydrology of overland flows and sediment transport. Instead of sampling over the entire hillslope, it is suggested that sampling should be concentrated over the ergodic length. The rill geometry is characterized in terms of easily measured quantities which are then used in the development of a stochastic averaging theory. This theory combines the hydrodynamics of the overland flow and erosion processes within the rills and on overland flow sections. These equations are solved numerically and the results are then compared to those obtained from hillslope experiments.

MATHEMATICAL FORMULATION

Sediment particles are removed (or deposited) at various time-space locations by the flow of water (resulting from rainfall) down the hillslope profile. This process involves detachment (dislodging of soil particles), transportation (entrainment and movement of soil particles with the surface flow) and deposition (when the transport capacity of the flow is reduced below that required for the existing suspended load). The active eroding agents in such situations are raindrop impact and overland flow.

Overland flow and sediment transport

The equations governing shallow water flow are derived from conservation of mass and linear momentum principles. The continuity equation for overland flow is (see, for example, Bennet, 1974):

$$\frac{\partial y}{\partial t} = i(x,t) - \frac{\partial q}{\partial x} = f_0(y;x,t) \quad (1)$$

where: $y(x,t)$ is the depth of water over the sediment surface (m); x is the spatial coordinate measured along the horizontal (m); t is time (s); $q(x,t)$ is the discharge per unit width of the flow ($\text{m}^2 \text{s}^{-1}$); and $i(x,t)$ is the net lateral inflow function (m s^{-1}). The momentum equation for overland flows may be modelled by the kinematic wave approximation given as:

$$S_f = S_0 \quad (2)$$

where S_f is the friction slope or slope of the total energy line and S_0 is the bed slope over which the flow takes place. The flow is completely specified by the use of a friction law relating the discharge and the depth at any time and at any

spatial location. A frequently used friction law for turbulent flows is given by the Chezy relationship:

$$q = Cy^{3/2}S_f^{1.2} \quad (3)$$

where C is Chezy's roughness coefficient. The process of erosion is presumed to occur so slowly that the change of the water profile with time is much faster than the corresponding change in the sediment profile. The conservation of mass equation for the sediment profile is given by:

$$\frac{\partial(cy)}{\partial t} + \frac{\partial(cq)}{\partial x} + (1 - \kappa) \frac{\partial y_s}{\partial t} = \frac{S_p}{\rho_s} \quad (4)$$

which may be rearranged in the form:

$$\frac{\partial c}{\partial t} = g_0(c, y; x, t) \quad (5)$$

where: $c(x, t)$ is the volumetric concentration of suspended sediment (vol/vol); κ is the soil porosity (fraction), $y_s(x, t)$ is the sediment elevation from a fixed datum (m); S_p is the detachment rate from rainfall impact ($\text{kg m}^{-2} \text{s}^{-1}$); and ρ_s is the density of the soil (kg m^{-3}). Bubenzer & Jones (1971) developed an expression for soil erosion due to raindrop impact as:

$$S_p = m(2.78 \times 10^{-7} I_r)^a k_e^b P_c^{-d} \quad (6)$$

where S_p is the soil detachment rate ($\text{kg m}^{-2} \text{s}^{-1}$); I_r is the rain intensity (mm h^{-1}); k_e is the total kinetic energy of rain drops (J m^{-2}); P_c is the percentage of clay in the sediment; $m = 1.5-3.0$; $a = 0.25-0.55$; $b = 0.83-1.49$; $d = 0.40-0.60$; and D is rainfall depth (mm). The expression for the kinetic energy of the raindrops is obtained as:

$$k_e = 24.16D + 8.73D \log \left[\frac{I_r}{25.4} \right] \quad (7)$$

The sediment continuity equation is by itself not sufficient to solve for the two unknowns, c and y_s , appearing in equation (4). We obtain another relation by considering a first-order reaction equation as proposed by Foster & Meyer (1975):

$$D_r = \sigma(T_c - q_s) \quad (8)$$

where: D_r is the erosion or detachment rate ($\text{kg m}^{-1} \text{s}^{-1}$); σ is the first-order reaction coefficient; T_c is the flow transport capacity ($\text{kg m}^{-1} \text{s}^{-1}$); and q_s is the sediment load ($\text{kg m}^{-1} \text{s}^{-1}$). Equation (8) states that the erosion (or deposition)

rate D_r is directly proportional to the difference between the sediment transport capacity and the sediment load at any time-space location. Therefore the surface flow erodes at a maximum rate when there is no sediment load. The coefficient σ in equation (8) is determined by:

$$\sigma = \frac{D_{rc}}{T_c} \quad (9)$$

where D_{rc} is the erosion capacity of the flow ($\text{kg m}^{-1} \text{s}^{-1}$). The flow transport and erosion capacities are defined as follows:

$$T_c = C_t(\tau - \tau_{cr})^{1.5} \quad (10)$$

$$D_{rc} = C_d \tau^{1.5} \quad (11)$$

where: $C_t = 0.001-0.5$; $C_d = 0.004-0.8$; τ_{cr} is the critical shear stress (kg m^{-2}); and τ is the unit tractive force over the flow bed (kg m^{-2}). The tractive force on the flow bed is given by:

$$\tau = \gamma y S_0 \quad (12)$$

where γ is the specific weight of water (kg m^{-3}).

Rill flow and sediment transport

The equations for flow and sediment transport within rills are similar to the overland flow equations and are mentioned here for completeness. For rectangular rills and using a kinematic wave approximation, the flow equation is as follows:

$$\frac{\partial h}{\partial t} = \frac{r(x,t)}{b(x,t)} - \frac{h}{b} \frac{\partial b}{\partial t} - \frac{C(S_0)^{1/2}}{b} \frac{\partial}{\partial x} \left\{ \frac{(bh)^{1.5}}{(b+2h)^{1/2}} \right\} = f_r(h, b; x, t) \quad (13)$$

and the equation for sediment transport within rills is (Foster, 1982):

$$\rho_s \frac{\partial(c_r A)}{\partial t} + \frac{\partial Q_s}{\partial x} = D_{rr} + D_L \quad (14)$$

where: $h(x,t)$ is the flow depth in the rill (m); $r(x,t)$ is the net lateral inflow per unit length ($\text{m}^2 \text{s}^{-1}$); $b(x,t)$ is the rill width (m); ρ_s is the mass density of the sediment particles (kg m^{-3}); c_r is the sediment concentration by volume; Q_s is the sediment load (kg s^{-1}); A is the flow cross-section area (m^2); D_{rr} is the erosion/deposition rate from the rill boundary per unit length ($\text{kg m}^{-1} \text{s}^{-1}$); and D_L is the lateral sediment inflow ($\text{kg m}^{-1} \text{s}^{-1}$). Equation (14) may be written in compact form as:

$$\frac{\partial(c_r)}{\partial t} = g_r(h, c_r; x, t) \quad (15)$$

Figure 1(a) shows the differences between the flow discharges within rills and on overland flow sections at the same downstream location on a hillslope (in Northern California, USA). Similarly, Fig. 1(b) shows that the sediment discharge within a rill is an order to magnitude greater than the sediment loss from overland sections. The results in these figures indicate the need for averaging based on the rill structure on the hillslope.

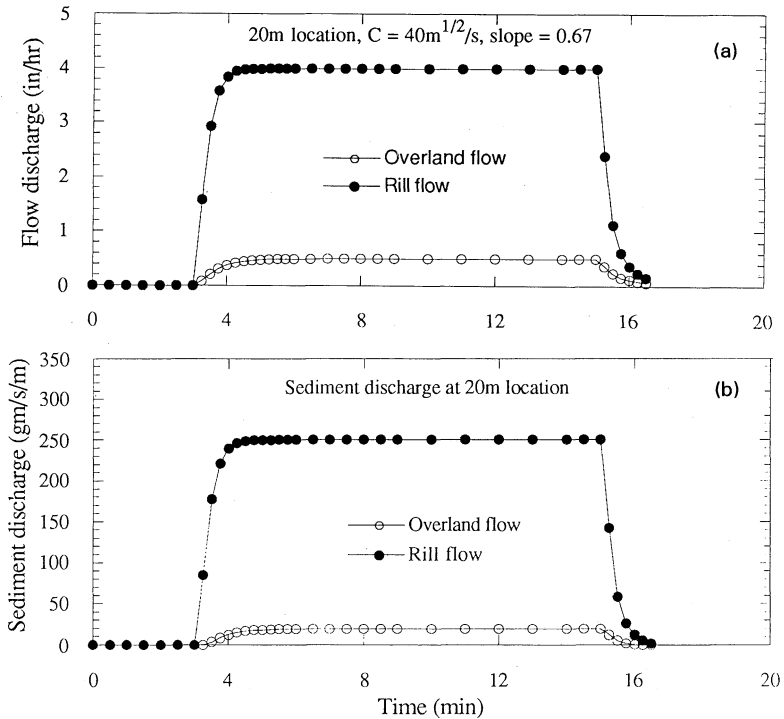


Fig. 1 (a) Flow discharges within a rill and over an adjacent overland flow. (b) Sediment discharges within a rill and over an adjacent overland flow section.

MODELLING OF THE RILL STRUCTURE

We first describe the important length scales over a rilled hillslope in a qualitative manner (c.f. Govindaraju & Kavvas, 1992). Over very small averaging intervals, any average property of the hillslope fluctuates very rapidly due to the highly irregular microtopography. We define the length scale, L_c , as the characteristic length scale of the microtopography. This length scale is a measure of the rapidity of the fluctuations in the rilled soil surface. As the averaging interval increases, the hillslope average property attains a

stable value. The scale at which this occurs is termed L_h and is the local spatial stationarity length scale. This stable value may change at very large values of the averaging interval due to changes in the terrain which induce large-scale spatial nonstationarity. The length scale L_r is likely to be greater than the width of an individual straight single hillslope which is the subject of this study. This large length scale L_r is more important for analysis at the watershed scale. The region where local spatial stationarity is applicable is termed the region of ergodicity and represents the ergodic length scale, provided that the correlation length of the property being investigated falls within this interval (i.e. correlation length $< L_h$). This is the only region where ergodicity is strictly valid. Within this study framework, ergodicity represents the equivalence of a spatial average representing a hillslope spatial property to the overall average of this property. Over the range of this length scale $[L_h, L_r]$, to be denoted by L_e , spatial averages may be used to represent the overall averages. As the averaging interval along the cross-slope (y coordinate in Fig. 2) increases, the surface tends to attenuate and the highly irregular microtopography which makes up the hillslope is gradually replaced by a more stable average at the ergodic length scale.

We are therefore trying to establish a continuum representation and to replace all the rapidly fluctuating spatial heterogeneity with integrated spatially averaged values. Thus, with a fine-grained mesh, the characteristic length scale of the microtopography L_c is important. However, using a coarser interval for averaging in space in the cross-slope direction leads to stable averages and to the existence of the ergodic length scale L_e for the mean behaviour of a particular hillslope property.

The magnitude of the averaging width at which ergodicity applies is of the utmost importance for the continuum representation. Conditions which need to be satisfied by the length scales are discussed briefly by Govindaraju & Kavvas (1992). The overland flow equations and the subsequent sediment transport equations are also affected by the continuum representation. Therefore a strategic averaging process needs to be chosen which reflects the physics involved at the microtopographic scale.

To estimate the characteristic length scale of the microtopography (L_c)

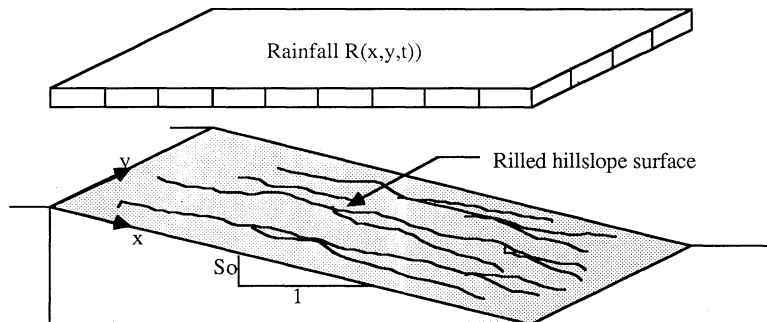


Fig. 2 Schematic sketch of a rilled hillslope surface.

and the ergodic length scale (L_h), we need to analyse the rill structure over the hillslope. We first define the rill indicator function (*RIF*). Consider a small interval Δy along any transect over the hillslope (see Fig. 2). Depending on the time-space location of this interval, it may or may not contain a rill. The *RIF* is defined as follows:

$$I(x,t;y) = \begin{cases} 1, & \text{if } (y, y+\Delta y) \text{ contains a rill} \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

Thus the *RIF* is an indicator function which takes on the value of zero or one depending on whether the interval it considers contains a rill or not. The characteristic length scale of the microtopography may be estimated by the following statistical criterion based upon the covariance function of the *RIF*:

$$\text{cov}[I(x,t;y), I(x,t;y+d)] = 0 \quad d > L_c \quad (17)$$

Thus L_c is a measure of the correlation length of the *RIF*. The smaller this length scale, the more irregular is the rilled hillslope surface. Two spatial locations which are separated by a distance greater than L_c are likely to be uncorrelated in terms of the rill structure.

Figure 3 shows the variation of the correlation of the indicator function in equation (16) as a function of the spatial lag, d , in equation (17). The correlation function in this figure is a normalized version of equation (16) and is expressed as:

$$\text{cor}[I(x,t;y), I(x,t;y+d)] = \frac{\text{cov}[I(x,t;y), I(x,t;y+d)]}{\text{var}[I(x,t;y)]} \quad (18)$$

The *RIF* is assumed to be spatially stationary at each hillslope transect (a consequence of the homogeneous hillslope assumption). The spatial location in these figures is measured from the top of the hillslope (e.g. Loc 40 implies 40 feet from the upstream location). All the cases considered in Fig. 3 indicate a

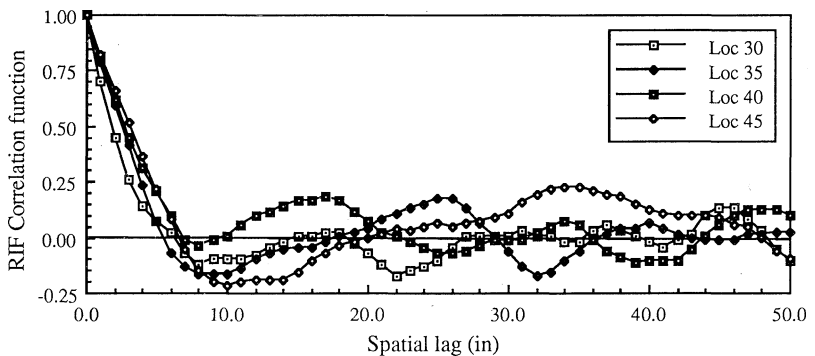


Fig. 3 Correlation function of the *RIF* at four different transects along an experimental hillslope in northern California, USA (1 in = 25.4 mm, 1 foot = 0.3048 m).

rapid decline in the correlation function. Once the spatial lag, d , in equation (17) increases beyond 0.25 m, the magnitude of the correlation function is bounded by 0.2 for the spatial locations in Fig. 3. Using similar data, Govindaraju & Kavvas (1992) estimated the characteristic length scale of the microtopography for this hillslope in California, USA as 1 foot (i.e. $L_c \approx 1$ foot). They identified other interesting properties of the rill structure through spectral representations of the rill indicator function.

We now address the issue of the ergodic length scale, which is the distance at which stabilization of any averaged property occurs. The expected spatial rill density (*ESRD*) is defined as follows:

$$\lambda(x,t;y) = \frac{1}{y} \int_0^y I(x,t;y) dy \quad (19)$$

Equation (19) shows that the *ESRD* $\lambda(x,t;y)$ is an average property over the distance y . From the definition of the *RIF* in equation (16), the *ESRD* in equation (19) represents the proportion of the transect that is occupied by rills. Therefore, the *ESRD* may be interpreted as the rill occurrence sample probability over the particular transect as a function of the hillslope averaging interval. Using the *ESRD* as the averaged property of interest, the stationary length scale may be evaluated using the following relationship:

$$\lambda(x,t;y) = \lambda(x,t) \quad y > L_h \text{ and } L_e > L_h \quad (20)$$

Equation (20) suggests that the ergodic length scale is that distance at which the *ESRD* does not change appreciably with an increase in the averaging distance.

Figure 4 shows the evolution of the expected spatial rill density (*ESRD*) along the hillslope at $x = 30.0, 35.0, 40.0$ and 45.0 feet. This figure shows that the expected spatial rill density achieves an almost stable value by 20 feet of averaging width. As is expected, the *ESRD* behaves very erratically during the first 10 feet of the averaging width. As the averaging width y increases to 20 feet, many of the fluctuations due to the rapidly varying microtopography have vanished, as enough rills have occurred within this width to stabilize the average. In Fig. 4, the stable values of the *ESRD* generally increase with increasing slope distance x from the upstream location. The ergodic length scale (L_e) was found to be 25 feet for the hillslope by Govindaraju & Kavvas (1992). Since the overland flow and sediment transport are directly influenced by the proportion of the hillslope occupied by the rills, it is likely that this ergodic length scale will hold for these hydrological phenomena also.

Let us denote the stable *ESRD* value by $\lambda(x,t)$ as in equation (20). This is obtained from $\lambda(x,t;y)$ for $y > L_h$. Therefore $\lambda(x,t)$ represents the space-time dependent average rill occurrence sample probability. It indicates the probability that the surface is being occupied by a rill over a hillslope width greater than L_h (but less than L_e). It may be noted that the *ESRD* can vary only between 0 and 1 because it expresses a proportion (see equation (19)). In soil

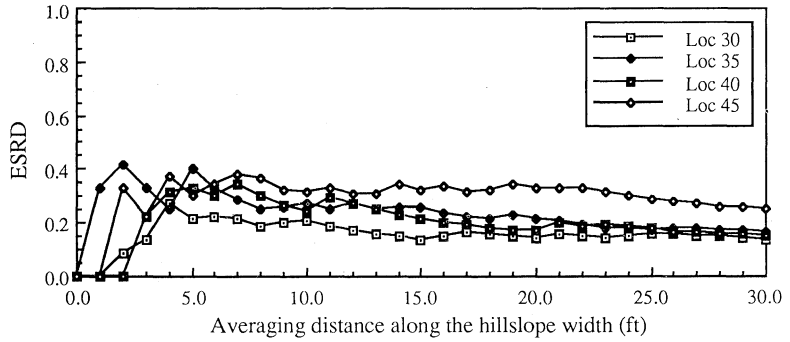


Fig. 4 Expected spatial rill density at four hillslope transects along an experimental hillslope in northern California, USA (1 in = 25.4 mm, 1 foot = 0.3048 m).

erosion research, the most common methods of predicting sediment discharge have used the effective water discharge, the effective critical shear stress or unit stream power to determine the transport capacity of the flow. All these quantities are related to the flow depth and velocity and they all increase with increasing slope distance from the top of the hillslope. It is therefore not surprising that the *ESRD* also increases in a similar manner.

HYDRODYNAMIC AVERAGING

Let us first see how the stochastic averaging theory applies to the overland flow equations. We want to describe the space-time behaviour of the mean flow depth $\underline{h}(x,t;\Delta y)$ which is obtained by averaging the overland flow and rill flow dynamics over a width Δy . The equation for the mean depth takes the form:

$$\frac{\partial \underline{h}(x,t;\Delta y)}{\partial t} = \langle f_r(h,b;x,t) \rangle \cdot \text{Pr}[\Delta y \text{ contains a rill}] + \langle f_0(y;x,t) \rangle \cdot \text{Pr}[\Delta y \text{ does not contain a rill}] \quad (21)$$

In the above equation Δy is the width of the slope over which the averaging process is being undertaken, the angular brackets denote the expectation operation, and f_r and f_0 are obtained from equations (1) and (13). The quantities in angular brackets reflect the need to take averages of the rill depths and widths and any other stochastic parameters in the overland and rill flow equations. The probabilities in equation (21) are evaluated using the *ESRD* and the statistics obtained for the indicator function in equations (19) and (20). When the averaging width Δy increases beyond the ergodic length scale L_h , then the probabilities in equation (21) do not change with Δy and are functions of only x and t .

The equations for sediment transport in the rills and the neighbouring

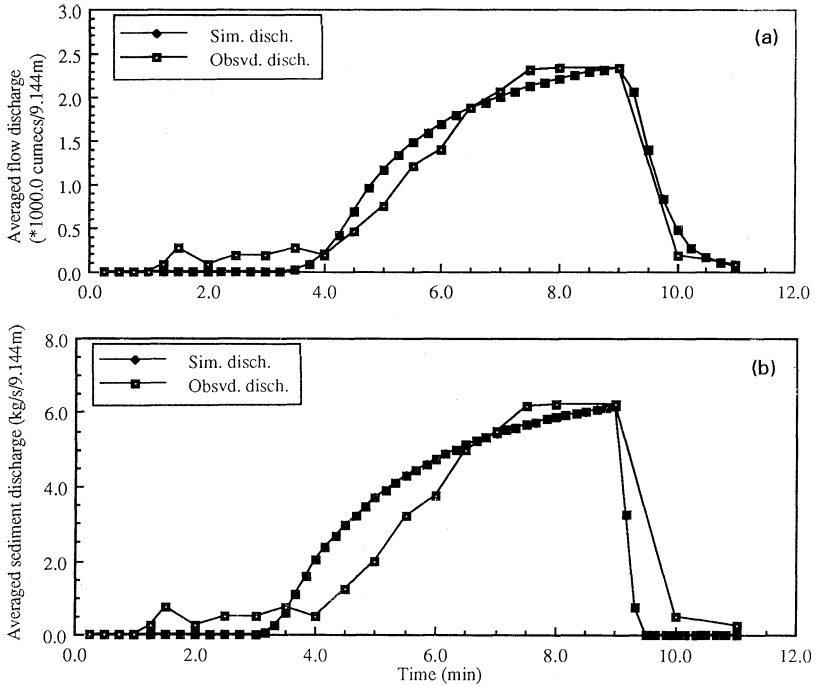


Fig. 5 Computed and observed flow discharges (a) and sediment discharge (b) for an experimental hillslope in California.

overland flow equations can be averaged in a similar manner. Let $\underline{c}(x,t;\Delta y)$ represent the hydrodynamically averaged concentration over the rill sections and the overland flow regions. The averaged sediment transport equation takes the following notational form:

$$\frac{\partial \underline{c}(x,t;\Delta y)}{\partial t} = \langle g_r(h_r, c_r; x, t) \rangle \cdot \lambda(x, t; \Delta y) + \langle g_0(h_0, c; x, t) \rangle \cdot [1 - \lambda(x, t; \Delta y)] \quad (22)$$

where g_0 and g_r are obtained from equations (5) and (15) respectively.

Equations (21) and (22) represent the stochastic averaging of the hydrodynamic equations of surface flows and sediment transport in rills and overland flow sections. The distributions of the rill flow depth and the overland flow depths can be obtained experimentally (as in Abrahams *et al.*, 1989; Govindaraju & Kavvas, 1992). Equations (21) and (22) are operator equations and their explicit representation shows the interdependence of the rill geometry and other stochastic influences on the averaged flow hydrodynamics. The performance of these equations against observed and simulated results over a hillslope in California, USA is shown in Fig. 5. These comparisons are satisfactory and indicate the utility of these equations in predicting average flow and sediment discharges over the hillslope.

FUTURE WORK

More experiments need to be performed to determine the dependence of the expected spatial rill density on geomorphological controls. The interaction of the rills and overland flow sections needs to be developed further to provide accurate coupling of the hydrodynamics for the two components.

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