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# A mathematical model of shear debris flow

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**Abstract** A description is given of a natural (Kanusyak mountain torrent, Carpathian Mountains) and of an anthropogenic (Baby Yar mountain torrent, Kiev) shear debris flow. Analytical and numerical methods of investigation of a St Venant-type dynamic model describing these natural phenomena are studied. The advisability is shown of a preliminary asymptotic solution of the modelling problem with the aim to determine the numerical modelling method, to reduce the amount of computations and to increase the efficiency of debugging the application program package. To check the reliability of the mathematical modelling, information is used about debris flow determined from its tracks.

## INTRODUCTION

Shear debris flows occupy a special place among natural and anthropogenic geodynamic and hydrological processes in mountains and gullied regions. They possess the highest denudation energy, are most destructive and often lead to losses of human lives. Such a shear debris flow took place in the Kanusyak mountain torrent in the Ukrainian Carpathian mountains in 1969 and an anthropogenic debris flow occurred in the Baby Yar mountain torrent in Kiev in 1961 and led to the death of 145 persons.

Because of the lack of an optimal monitoring system, data on the dynamic characteristics of debris flows are incomplete. Only morphometric (geometrical) characteristics of the flows are determined with useful accuracy. This necessitates the application of special methods to determine debris flow characteristics, one of the methods being mathematical modelling based on a hydraulic scheme developed by Grigoryan (1979), Mironova & Eglit (1988), and Danilova & Eglit (1977). The model enables a set of time-varying characteristics to be obtained in any section of the transit zone: depth, cross-sectional area, velocity and discharge of debris flow.

Data on shear debris flows and the basic results of their mathematical modelling are presented in the paper.

†We regret to announce the death of Dr Yablonskiy in February 1992.

## BASIC DATA ON SHEAR DEBRIS FLOWS

The Kanusyak mountain torrent is located on the slope of the mountain with the same name in the Gorgany mountain ridge at a height 800-1200 m above sea level. The source area of the debris flow is formed by strongly weathered flysch sandy loam deposits. A source area with a volume of about 70 000 $m^3$ was formed by the shearing of a soil block on 8 June, 1969 following a rainfall of more than 200 mm in 24 h. The length of the source was about 230 m, the slope of the bed in the source area was about 20-28°. The debris flow comprised a mixture of lumps of soil, stones, gravel, sand, mud and clay. Fine earth washed from the surface of the basin formed a gliding surface at a depth about 10 m along which the block of soil forming the debris flow slid off. Water runoff took place on the same surface during the storm.

The transit reach is about 630 m long. Its slope is 18-20°. The bed consists mainly of thick-layered poorly destructible sandstone. The channel cross section is trapezoidal. To compute the characteristics of the debris flow, a section located at the end of the transit reach was selected. The dimensions of the flow section are as follows: maximum depth of the flow is 8 m, width at the top is 17 m, the area of the cross section is 84 $m^2$. The slope of the flow surface in the transit reach is equal to the slope of the bed.

Debris deposition begins 30 to 40 m downstream from the control section. The deposits have a complex shape: they fill the offing of the flow 100 m in length and also form the debris cone at the bottom of the valley into which debris flows. The maximum depth of deposition at the vertex of the cone is 16 m. The volume of the debris flow deposit was about 70 000 $m^3$.

The Baby Yar Ravine is located on the outskirts of Kiev. Hydraulic emplacement of soil was carried out in the upper part of the ravine via a pipeline in the postwar period (1950-1961). Altogether about 3.5 million $m^3$ of strongly moistened sandy and loamy soils was stored from 125 to 165 m above sea level. In the morning of 13 March 1961, the earth-fill dam which was holding the soil failed and about 600 000 $m^3$ of soil began to move. A flow of mud mixture traversed a path of about 1600 m along the ravine bed. Buildings and houses were destroyed, an area of the city of about 25 ha was blocked with deposits. The maximum depth of deposition was about 4 m.

A control section was located at a distance of 1440 m from the downstream boundary of the source area. The maximum depth of the flow in this section was 7 m, the cross-sectional area was 420 $m^2$. The slope of the transit part of the bed was about 0.57-0.37°. The cross section of the flow was trapezoidal.

Data have been prepared for both shear debris flows characterizing distribution of the amount of friable fragmental material (debris) along the bottom of the source area, the morphometry of the path, soil density in the source area and in the state of motion. For the Kanusyak torrent the latter was taken to be equal to 2300 kg $m^{-3}$ and for Baby Yar 1500 kg $m^{-3}$.

Maximum flows of debris of 554 $m^3$ $s^{-1}$ (Kanusyak) and 2200 $m^3$ $s^{-1}$ (Baby Yar) have been computed using Yablonskiy's mean velocity calculation formula.

The time distribution of depths, cross-sectional areas, velocities and volumetric discharge of debris flows for control sections have been obtained from the mathematical model of debris flow motion described below. The modelling has made it possible also to carry out the analysis of possible shear mechanisms and to determine in which part of the source area (upstream or downstream part) the loss of soil stability occurred and in what succession the elements of the source area began to move: beginning from the upstream border of the source area or from the downstream one.

## STATEMENT OF THE PROBLEM OF MODELLING DEBRIS FLOW IN A BED WITH TRAPEZOIDAL CROSS SECTION

A hydraulic model of the St Venant type described by Danilova & Eglit (1977) assuming a constant density of debris flow mixture in all parts of the flow is used to describe the motion of debris flow. The mass and momentum conservation equations have the form:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Av) = q \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = g\sin\alpha - \frac{1}{A}\left[F_1 v + \frac{\partial F_2}{\partial x} - F_3\right] \tag{2}$$

where $t$ is time, $x$ is distance along the bed, $A$ is cross-sectional area of debris flow, $v$ is mean cross-sectional velocity of the flow along the $OX$ axis, $q$ is volume of mass deposited by debris flow per a unit of length in a unit time, $g$ is acceleration of gravity, $\alpha$ is angle of bed slope with respect to horizon, $F_1$, $F_2$, $F_3$ are parameters associated with friction, change of level in the flow, pressure from side walls of the bed.

For a bed with trapezoidal cross section, $A$, $F_1$, $F_2$, $F_3$ can be written as follows:

$$A = bh + 0.5h^2(\tan\beta_1 + \tan\beta_2)$$

$$F_1 = \begin{cases} \kappa|v|\ell + 0.5\mu gh(b+\ell)|v|^{-1}\cos\alpha, & h \le h^* \equiv \tau^*(\rho\mu g\cos\alpha)^{-1} \\ \kappa|v|\ell + 0.5\mu gh^*[2\ell - h^*(\cos^{-1}\beta_1 + \cos^{-1}\beta_2)]|v|^{-1}\cos\alpha, & h > h^* \end{cases}$$

$$\ell = b + h(\cos^{-1}\beta_1 + \cos^{-1}\beta_2) \tag{3}$$

$$F_2 = \frac{1}{2}gh^2\cos\alpha b + \frac{1}{6}gh^3\cos\alpha(\tan\beta_1 + \tan\beta_2)$$

$$F_3 = \frac{1}{2}gh^2\cos\alpha\frac{\mathrm{d}b}{\mathrm{d}x} + \frac{1}{6}gh^3\cos\alpha\frac{\mathrm{d}}{\mathrm{d}x}(\tan\beta_1 + \tan\beta_2)$$

where $b$ is the width of the bed bottom, $h$ is the depth of the flow along a perpendicular to the bed bottom in its axial point, $\beta_1$, $\beta_2$ are the angles between the perpendicular and the walls of the bed, $\kappa$, $\mu$ are coefficients of hydraulic and dry friction, $\ell$ is the wetted perimeter of the bed, $\tau^*$ is limiting value of dry friction stress, $h^*$ is the height of the flow at which dry friction stress reaches the value of $\tau^*$ as shown by Grigoryan (1979), $\rho$ is the mean value of debris mixture density. The path of a debris flow can be conventionally divided into three zones: initiation, transit, deposition. A flooded block of friable soil whose shear has caused debris flow is located in the initiation zone. The transit part for shear debris flows is characterized by debris flow acceleration at an insignificant (negligible) increment of mass (volume). The deceleration of debris flow and its complete stopping comes about in the deposition zone, i.e. $q \neq 0$ in the mass conservation equation (1) in this zone whereas $q \equiv 0$ for the first two parts.

Debris flow motion is considered under the following conditions: no soil inflow takes place at the trailing edge of the flow

$$A = 0, \; v = 0, \; x = 0 \tag{4}$$

incorporation of soil is also absent at the leading edge of debris flow

$$A = 0, \; v = w \tag{5}$$

where $w$ is the velocity of the leading edge of the debris flow.

Conditions (4) and (5) are characteristic of a pure shear process.

## ASYMPTOTICAL SOLUTION OF ST VENANT EQUATIONS FOR DEBRIS FLOWS IN A BED WITH TRIANGULAR CROSS SECTION

Let us consider the asymptotical solution of equations (1)-(5) for a debris flow in a bed with triangular cross section with constant geometrical parameters with

$$\begin{aligned} &q = 0, \;\; b = 0, \;\; h < h^*, \;\; F_3 = 0 \\ &\alpha(x) = \alpha, \;\; \beta_1(x) = \beta_1, \;\; \beta_2(x) = \beta_2 \end{aligned} \tag{6}$$

The asymptotical method of solving equations (1) and (2) involves a change-over from equations (1) and (2) to kinematic wave equations by Lighthill & Whitham (1955). The latter are obtained from equation (1) and simplified equation (2) in which the differential terms are eliminated and they have the following form according to (3), (6):

$$\begin{aligned} &\frac{\partial h^2}{\partial t} + \frac{\partial}{\partial x}(h^2 v) = 0, \quad v = f\sqrt{h} \\ &f = \sqrt{(gI\cos\alpha)/2\kappa}, \; I = [\sin(\beta_1 + \beta_2)\tan\alpha/(\cos\beta_1 + \cos\beta_2)] - \mu \geq 0 \end{aligned} \tag{7}$$

Using the method of characteristics by Bakhvalov & Eglit (1973) for the solution of (7) and assuming that the whole mass of debris is concentrated in point $x = 0$ at time $t = 0$ (Hunt, 1984), we obtain:

$$h = 16x^2/25f^2t^2, \quad v = 4x/ty, \quad 0 \leq x \leq \eta \tag{8}$$

where $\eta$ is the coordinate of the leading edge of the debris flow.

Solution (8) is correct if a debris flow has a sufficient extension along its axis and the parameter distribution in it is such that the mean velocities and depths of the flow vary smoothly (Woolhiser & Ligget, 1967) and also the stability condition of wave solution (8) is met (Whitham, 1958) which has the form:

$$16\kappa \geq I \geq 0 \tag{9}$$

It follows from (8) that the height of the flow is maximum at its leading edge $h = h_\eta > 0$, $x = \eta$ but according to (5) $h = 0$ at $x = \eta$. Thus, there is a break at point $x = \eta$ whose structure is described by the solution of complete equations (1) and (2) smoothly changing $h$ from $h_\eta$ to zero (Galin, 1959). This solution has the form:

$$\begin{aligned} -\xi &= \sin(\beta_1 + \beta_2)[h + h_\eta \ln(1 - h/h_\eta)]/[\sin(\beta_1 + \beta_2)\tan\alpha \\ &\quad - \mu(\cos\beta_1 + \cos\beta_2)] \\ v &= w, \quad h_\eta = w^2f^2, \quad \xi = x - wt \end{aligned} \tag{10}$$

In constructing (10), it was assumed that the motion in a narrow zone corresponding to a break of wave solution (8) is stationary in the system of coordinates $\xi = x - wt$ moving integrally with the leading edge of the flow at a velocity $W$ (Bakhvalov & Eglit, 1973).

Coordinate $\eta$ and velocity $w$ of the leading edge of debris flow are determined from the condition of constancy of volume:

$$\begin{aligned} &\int_0^\eta A\,\mathrm{d}x = V \\ &\eta = 5/2[4V/(\tan\beta_1 + \tan\beta_2)]^{1/5}(ft)^{4/5} \end{aligned} \tag{11}$$

$$w = \frac{\mathrm{d}\eta}{\mathrm{d}t} = 4\eta/5t \tag{12}$$

Equations (8)-(12) are asymptotic solutions of equations (1) and (2) with conditions (3)-(6).

The relationship between the maximum area of the flow cross section in a particular section and debris volume $V$ follows from formula (11):

$$A_\eta = 5V_\eta^{-1} \tag{13}$$

It is seen from formula (13) that $A_\eta$ (and therefore $h_\eta$) do not depend on coefficients of dry and hydraulic friction and are determined only by the volume of debris flow mixture and by the location of the control section.

Now let us assume that the motion of the real debris flow takes place in a bed with a triangular cross section and with constant geometric parameters and all conditions are met at which the asymptotic solution is constructed. Then it follows from (13) that at given $A_\eta = 84$ m$^2$, $\eta = 630$ m and $A_\eta = 420$ m$^2$, $\eta = 1440$ m the value $V$ is equal to 10 584 m$^3$ for Carpathian and 120 960 m$^3$ for Kiev debris flows, respectively, which makes up about one fifth of the whole volume of debris mixture of each flow. Therefore, the initiation into motion of debris forming soil in each of the bases considered above cannot proceed from the top downwards along the valley profile. It proceeds from below upwards in the form of separate portions.

The presented asymptotic solution of St Venant's equations in view of the idealization of the conditions does not pretend to be a quantitative description of the complex natural process of debris flow but it can serve as some prompt for the development of the general plan of numerical simulation of debris flow using computers.

## RESULTS OF NUMERICAL SIMULATION OF DEBRIS FLOWS

The program package LAVINA based on equations (1)-(5) was used for the numerical simulation of shear debris flows (Mironova & Eglit, 1988). Problems have been solved of reconstruction of debris flow motion from the limited observed data available.

Only a few versions of numerical computations have been required to confirm the fact that the mathematical model should be accepted as the final one in which, as in the asymptotic solution, the loss of stability of the friable material of debris flow source area begins in its downstream part. This common character of the asymptotic and numerical methods manifests itself especially clearly for debris flow in Baby Yar.

Generally, a special feature of the numerical model algorithm is the check for the equality of the computed values of the area of the flows cross sections and the volumes of the debris flow to the values measured under field conditions by the traces of debris flow. Besides, the agreement was estimated between the maximum discharge and the maximum discharge determined using Yablonskiy's formula for velocity computation.

A selection of numerical values of mathematical model parameters $\eta$, $\kappa$, $\tau^*$, $\tau^*/\rho$ was carried out with regard to the available data on the physical properties of debris mixtures as well as with regard to dimensions of sections of formed flows. Thus, the value of coefficient $\eta$ characterizing the impact of dry (Coulomb's) friction was taken to be approximately equal to the tangent of the slope angle of the surface of undisturbed debris deposits. For the Kanusyak debris flow, it was finally taken that $\eta = 0.026$ and for the debris flow in the Baby Yar torrent $\eta = 0.001$. Parameter $\kappa$ characterizing hydraulic friction is equal to 0.1 for the Carpathian debris flow and 0.03 for Baby Yar.

In the real debris flows being considered, the motion of the debris mixture was carried out without considerable destruction of soils along the beds of the flows, therefore it is correct to take into account the influence of dry friction stress $\tau^*$ in equations (1)-(3) combined with the value of debris density $\rho$ in the form of parameter $\tau^*/\rho$. The calculations have shown that this parameter is insignificant for these debris flows. This is associated with the fact that the depths of formed flows have been much less than $h^*$. As in asymptotic solution, a variation of the values $\mu$ and $\eta$ has practically no impact on the values of flow depths. The value of debris flow velocity depends substantially on these parameters.

Figures 1 and 2 illustrate the variation of the main characteristics of debris flows for control sections obtained as a result of the numerical simulation. Of interest is a regular decrease of the maximum depth of the flow in sections as we move away from the source area in Baby Yar mountain torrent which is shown in Table 1.

Thus, the numerical simulation has shown a clear picture of a spreading debris wave in the process of motion which also agrees with the results of the analytical solution by formula (13).

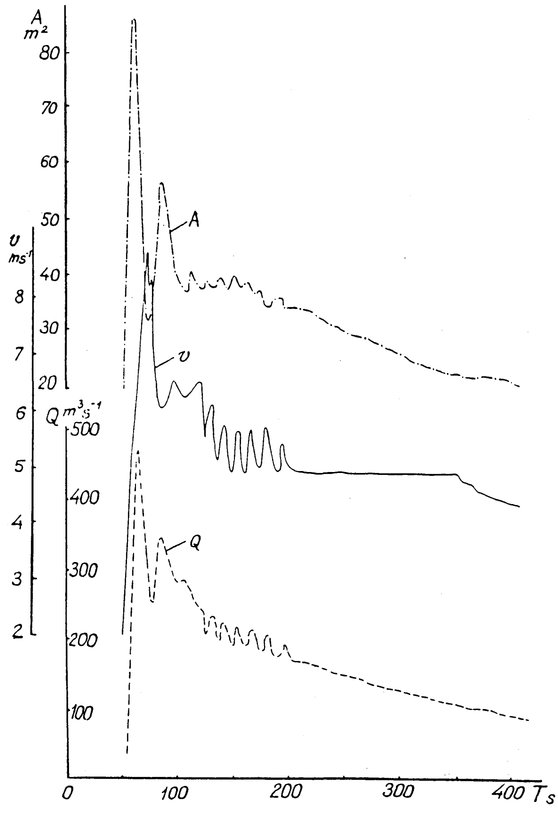


**Fig. 1** Variation of cross-sectional area $A$, mean cross-sectional velocity $v$ and volumetric discharge $Q$ of the Kanusyak debris flow (Carpathians) with model parameters $\mu = 0.260$, $\kappa = 0.100$, $\tau^*/\rho \geq 30\ m^2\ s^{-2}$.

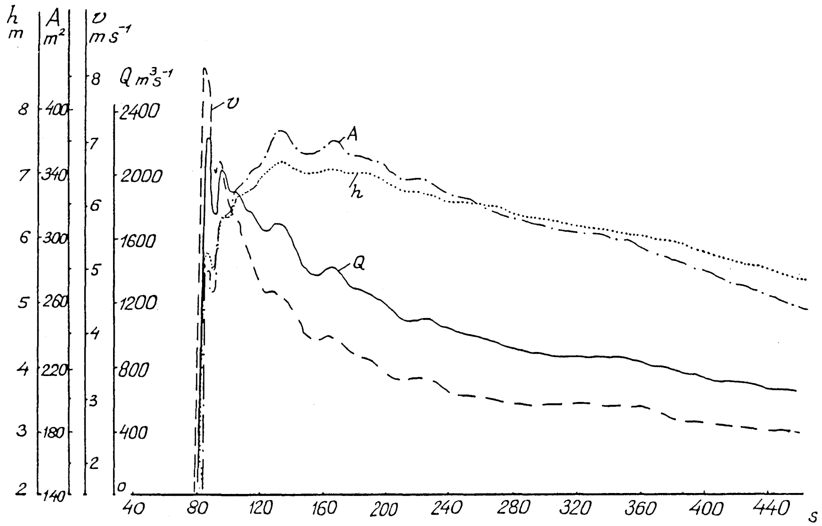


**Fig. 2** Variation of depth $h$, cross-sectional area $A$, and mean cross-sectional velocities $v$ of the Baby Yar debris flow on 13 March 1961 with model parameters $\mu = 0.001$, $\kappa = 0.030$, $\tau^*/\rho \geq 10$ m$^2$ s$^{-2}$.

## FUTURE WORK

In the future, the mathematical modelling of debris flows should be extended to the process of debris mixture deposition. Such solution should be represented by a two-dimensional model taking into account the spreading process of the debris mixture over sites with small slopes or over horizontal sites. The LAVINA application program package makes it possible to carry out such computations.

Of interest is the use of the results of the mathematical modelling of debris flows in real debris flow forecasting systems with the application of automated systems for the collection and processing of current information on the state of debris flow basins.

## CONCLUSION

Mathematical modelling of the motion of natural catastrophic processes of

**Table 1** Values of maximum depth of debris mixture flow computed with model parameters $\mu = 0.001$, $\kappa = 0.030$, $\tau^*/\rho \geq 10$ m$^2$ s$^{-2}$.

| Distance of cross section from top of source area $x$ (m) | Maximum depth of flow $h$ (m) |
|---|---|
| 440 | 13.52 |
| 620 | 13.18 |
| 820 | 9.97 |
| 1220 | 9.42 |
| 1440 | 7.16 |

avalanches in nature makes it possible to obtain important information about the time variation in different sections of the main flow characteristics: depth, velocity, flow rates which are very difficult (and often impossible) to observe and measure in a natural situation.

Combined analytical (asymptotical) and numerical modelling methods mutually enrich each other. Asymptotical methods should be used at the first stage of the modelling which will substantially accelerate the debugging of the application program package.

Taking into consideration the urgent need for debris flow monitoring systems comprising automated processes of observation, collection and processing of information and the forecasting of the state of debris objects, mathematical modelling makes an important contribution to the problem of improving the warning of dangerous phenomena.

## REFERENCES

Bakhvalov, N. S. & Eglit, M. E. (1973) Investigation of uni-dimensional motion of an avalanche along a flat slope (in Russian). *Izv. AN SSSR Mekhanika Zhidkosti i Gaza* **5**, 7-14.

Danilova, E. M. & Eglit, M. E. (1977) Motion of channelled avalanches (in Russian). *Materialy Glaciologicheskikh Issledovaniy* **31**, 65-74.

Galin, G. Ya. (1959) To shock wave theory (in Russian). *Dokl. AN SSSR* **127**(1), 55-58.

Grigoryan, S. S. (1979) A new law of friction and mechanism of large-scale avalanches and landslides (in Russian). *Dokl. AN SSSR* **244**(4), 846-849.

Hunt, B. (1984) Perturbation solution for dam break floods. *J. Hydraul. Engng* **110**(8), 1058-1071.

Lighthill, M. J. & Whitham, G. B. (1955) On kinematic waves flood moment in long rivers. *Proc. Roy. Soc.* ser. A, **229**(1178), 281-316.

Mironova, E. M. & Eglit, M. E. (1988) Application program package for numerical simulation of avalanches, debris flows and water flows (in Russian). *Materialy Glaciologischeskikh Issledovaniy* **63**, 161-165.

Whitham, G. B. (1958) Some comments on wave propagation and shock wave structure application to magnetohydrodynamics. *Communs. on Pure and Appl. Math.* **12**(1), 113.

Woolhiser, D. A. & Ligget, J. A. (1967) Unsteady unidimensional flow over a plane - the rising hydrograph. *Wat. Resour. Res.* **3**(3), 753-771.