A review of basic snow mechanics

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Abstract. In an *introductory* section the subject is defined, research goals are identified, a historical perspective of snow mechanics is outlined, and requirements for theoretical analysis, in the form of constitutive equations and failure criteria, are specified.

A section on *deformation* opens with a review of idealized rheology for nondestructive loading, giving general differential equations and pertinent special solutions for one-dimensional models. Simple elastic properties, as derived from quasistatic and dynamic tests on snow, are summarized, and effects of rate, frequency, density, temperature, etc. are considered. Viscoelastic properties are described in terms of complex moduli, viscosity for creep transients, loss factor, etc., and independent variables such as temperature, density and frequency are taken into consideration. Viscous behaviour in sustained creep is introduced, and complications of characterization are emphasized during presentation of data obtained by measurements involving uniaxial stress and uniaxial strain. Simple stress analysis is introduced to facilitate further discussion; points covered include separation of stress and strain tensors into deviatoric and dilatational components, definition of stress and strain invariants, shear and bulk viscosities, viscous analogue of Poisson's ratio, and determination of stress components from equilibrium equations. Volumetric strain and irreversible compressibility are discussed, and 'equation of state' characterization is proposed. Pressure/density data for uniaxial stress states are compiled, and a plot of bulk stress against specific volume is developed for various rates and temperature conditions. Deviatoric stress/strain-rate relations for conditions of negligible volumetric strain rate are considered, drawing on data derived from unconventional sources, including uniaxial compression strength tests and creep profiles for settled snow slopes. The section closes with a discussion of the problem of formulating general constitutive equations for multiaxial stress states.

A section on *failure* starts with discussion of failure, strength, and failure criteria, especially as the terms apply to snow. Strength measurements in uniaxial tension and compression are considered, and available data are compiled. Definition of shear strength is considered, the merits and limitations of c–ϕ characterization of snow are discussed, and some of the more reliable shear strength data are compiled. Physical and statistical failure theories based on structural defect concepts are mentioned, empirical relationships between the strength of snow and its porosity are described, and the possibility of deriving a relation between ϕ and porosity is introduced. Attempts to develop phenomenological failure criteria for snow are mentioned briefly. A cautionary note on discontinuous collapse of snow under compression is included, and correlation of arbitrary strength indices with basic strength properties is discussed. Correlation of strength and elastic modulus is explored as a possible method for estimating strength from nondestructive *in situ* tests.

A section on *boundary friction* introduces the physical theory relating to kinetic friction of coherent snow surfaces, and summarizes experimental findings that give variations of kinetic friction with contact pressure, dimensions of contact areas, sliding speed, ambient temperature, and slider materials. Static friction and adhesion are distinguished and discussed, and methods for measuring or estimating true static friction are suggested. Boundary friction of fluidized snow is considered, and drag induced by inertial effects in a well-diffused turbulent boundary layer is treated in an exploratory calculation.

The review *concludes* with an appraisal of the major difficulties in snow mechanics and with some suggestions for research direction.

INTRODUCTION

Scope and trends

Narrowly defined, snow mechanics deals with forces and displacements in all forms of snow, i.e. with the kinematics, dynamics and energetics of snow in both condensed and dispersed states. More broadly defined, it also embraces the underlying physics of processes relevant to mechanical behaviour, and the useful but disconnected empiricism associated with snow engineering, avalanche prediction, etc. Since snow mechanics evolved as an identifiable field of study about 40 years ago, a good deal of attention has been given to the latter topics, but work on basic mechanics has tended to be more sporadic. Systematic application of formal mechanics to problems concerning stress/strain/time relations in complicated physical situations has been rather slow to develop, which is hardly surprising, since snow has very complex properties, exhibiting nonlinear viscoelasticity and undergoing large strains, both volumetric and deviatoric, under typical loading conditions. Because snow normally exists at homologous temperatures above 0.9, its properties are highly temperature-sensitive, and even its basic structure is thermally unstable. As a consequence of these things, it has been impractical to formulate general constitutive equations and failure criteria that are both realistic and tractable.

Some current trends suggest that this situation might soon change for, in principle, the powerful formalism of modern rational mechanics is capable of dealing with the complicated behaviour of snow. However, it seems unrealistic to expect early solution of significant boundary value problems if very complex constitutive equations are employed.

While attempts at highly rigorous theoretical analysis are obviously worthwhile, it is necessary to keep sight of the fact that snow mechanics is essentially a practical science, in which useful results are of overriding importance. Thus, the main thrust of research ought to be concerned with progressive upgrading of methods needed for the accomplishment of practical tasks. Such an approach demands some compromise between the almost impossible demands of rigorous theoretical analysis and the confusions of wholly empirical procedures.

Perhaps the most troublesome single characteristic of snow is its high compressibility, which makes it behave very differently from most solids and granular materials. As long as the bulk stress in a multiaxial stress field is below a certain critical value, the deformation and rupture of snow can be treated in much the same way as for low compressibility solids, but when this critical value is exceeded the snow undergoes large and irreversible volumetric strain, acquiring vastly different mechanical properties. Simple treatment of compressibility could well be one of the major goals for theoretical research.

In this short review a complete coverage of snow mechanics is not feasible, and so emphasis is given to the basic mechanics of dry coherent snow (the major omissions are wet snow and boundary value problems). The chief aim is to summarize the mechanical properties of snow within the context of established continuum mechanics, and to explore the formulation of manageable constitutive equations and failure criteria.

Historical perspective

Systematic study of snow mechanics began in the 1930s, and in a remarkably short time most of the main features of the stress/strain/time relations for coherent snow were broadly defined by Swiss investigators. The constitutive relations and failure criteria considered for solution of boundary value problems were necessarily simple ones that did not fully reflect the known behaviour of snow, but basic body-force problems in the avalanche field tended to be of a type in which internal stresses are determined largely by the boundary conditions and the self-weight of the material, almost irrespective of rheologic properties. Simple failure criteria of the Coulomb type were adequate for treating planar slip. More detailed knowledge of snow properties was used to refine results by accounting for creep effects and variations due to temperature, density, snow type, etc.

Interest in snow mechanics widened considerably about 20 years ago, stimulated largely by international research in Antarctica and by military activity in Greenland.

The main emphasis shifted from stability and rupture of snow slopes to the long-term creep of snow under relatively steady sustained loadings, as in densification of ice cap snow, foundation settlement, tunnel closure, and structural loadings. In the latter type of problem initial transients of deformation could often be ignored, and simple viscous analyses became useful, although it was always accepted that assumptions of newtonian viscosity were gross oversimplifications. In some cases, complexities introduced by the rheology of snow were sidestepped by replacing continuum mechanics with special analogue models, e.g. in foundation studies. Another class of problems involved the bearing capacity and shear resistance of snow surfaces, especially as they affect the sinkage and traction of vehicles. In some of these problems simple elastic analyses sufficed (e.g. Boussinesq analysis for a high-speed wheel on dense snow) and ample data on elastic moduli were obtained from field seismic studies and laboratory tests. However, in the 'trafficability' studies there was little attempt to define limit conditions by formal failure criteria; the emphasis was on arbitrary strength indices that could be measured easily in the field.

In the past decade the general scope of snow mechanics was further expanded by formal hydrodynamic consideration of dispersed snow as a two-phase flow (as in blowing snow, dust avalanches, snowplough discharges, etc.), and by study of intense compression and shock propagation (as in 'equation-of-state' problems involving explosions, impacts, and projectile penetration). One important hydrodynamic aspect that has not yet received much attention is the problem of snow as a fluidized solid (e.g. snow under slipping vehicle tracks, on a snowplough blade, or in a dense avalanche).

The current situation, at least as it appears from the United States, is that interest in the mechanics of deposited snow has waned somewhat with the decline in operations on polar ice sheets, and most work is again motivated by avalanche protection requirements. In both the US and Europe there are signs of rising interest in applying modern continuum mechanics, which permits maintenance of a high degree of analytical generality for complicated constitutive equations, but at the same time practical problems are still being tackled largely by empirical methods.

In some respects it seems that present-day activities are concentrated at opposite ends of the research spectrum, with a tendency for neglect of the middle ground. Field investigators and experimentalists are sometimes forced to operate without much framework of theoretical guidance, while theoretical mechanicians may be frustrated by lack of suitable comprehensive data on material properties. There seems to be a definite need for orderly assessment of snow properties, for realistic but simplified mechanical characterization of snow, and for sound and workable theory that can be applied to boundary value problems.

Requirements for theoretical analysis

In its simplest form, coherent deposited snow can be regarded as a quasihomogeneous sintered compact of equant ice grains. As such, it possesses most of the mechanical properties of solid ice, i.e. it displays temperature-sensitive nonlinear viscoelasticity under typical loading conditions, and rupture may occur by viscoplastic yielding, by brittle fracture, or by a transitional process. However, snow is highly porous and it has high, and largely irreversible, compressibility. It is this high irreversible compressibility of snow that distinguishes it from ice and most other engineering materials.

The basic requirements for formal mechanical analysis are *constitutive equations* that relate stress, strain and time for multiaxial stress states, and *failure criteria* that give the critical relationships of principal stresses at yield or rupture. Ideally, these should be general equations that permit the tracing of stress or strain history from an initial reference state, and they should be capable of accounting for temperature variations.

In the case of solid ice, there is now sufficient information on which to base constitutive equations for elastic deformation and constant-rate flow, but it is not yet possible to derive failure criteria that are applicable outside the range of viscoplastic yielding (ice is apparently anomalous with respect to 'Griffith-type' materials). However, the rheological properties of ice can already be expressed in forms that are becoming too complicated for useful application in practical continuum mechanics. Looking at snow, which is a much more complicated material than solid ice, it appears that presently available data are inadequate for defining general constitutive equations or failure criteria. Once this deficiency is remedied it will be possible to define the required relationships, but in general form these equations will almost certainly be too complicated for useful application in boundary value problems.

While general description of rheologic properties and failure conditions is obviously a desirable research goal, short-term needs in applied mechanics will probably have to be met by piecemeal simplifications appropriate to particular problems. In some cases these simplifications are straightforward (e.g. purely elastic problems), but in others there are still difficulties in arriving at valid representations (e.g. in problems involving simultaneous volumetric and deviatoric strains).

DEFORMATION

Idealized rheology for nondestructive loading

Standard rheological models (Fig. 1) are widely used for describing the mechanical behaviour of snow. They are convenient for qualitative descriptions, and they provide a standard terminology, but because the elasticity and viscosity of snow are strongly nonlinear they have very limited quantitative value. Nevertheless, since some of the reported viscoelastic properties of snow are derived by assuming certain models and analysing test data accordingly, it is necessary to understand these models.

Burgers model

The deformation of snow and ice under nondestructive loading has long been described in terms of the four-element Burgers model. The general equation giving the rheologic response of this model is:

$$\ddot{\sigma} + \left(\frac{E_M}{\eta_M} + \frac{E_M}{\eta_K} + \frac{E_K}{\eta_K}\right)\dot{\sigma} + \frac{E_M E_K}{\eta_M \eta_K}\sigma = E_M \ddot{\epsilon} + \frac{E_M E_K}{\eta_K}\dot{\epsilon} \tag{1}$$

where σ and ϵ are stress and strain ($\dot{}$ and $\ddot{}$ denoting first and second time derivatives), E_M and η_M are modulus and viscosity for the Maxwell unit, and E_K and η_K are modulus and viscosity for the Kelvin–Voigt unit. Solutions of special interest are those for abrupt application or removal of constant stress, and for application of constant strain rate or constant stress rate.

For constant stress σ_0 applied abruptly at $t=0$, the required solution is:

$$\epsilon = \sigma_0\left\{\frac{1}{E_M} + \frac{t}{\eta_M} + \frac{1}{E_K}\left[1 - \exp\left(-\frac{E_K}{\eta_K}t\right)\right]\right\} \tag{2}$$

Equation (2) gives a strain–time relation, or creep curve, for the standard creep test. The curve shows instantaneous elastic strain, followed by decelerating (primary) creep, which in turn is succeeded by creep at constant rate (secondary creep). It does not simulate tertiary creep. The various terms of equation (2) are easily identified with the elements of the model; in the exponential term (η_K/E_K) is the *relaxation time,* i.e. the time required for strain to decay to $1/e$ of its initial value when load is removed.

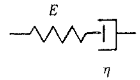
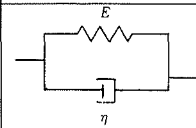
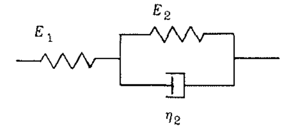
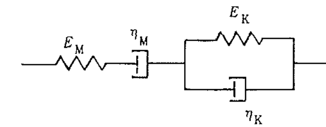
Model		Maxwell	Kelvin–Voigt	3–Element	Burgers
Diagram					
General equation		$\dot{\epsilon}=\frac{1}{E}\dot{\sigma}+\frac{1}{\eta}\sigma$	$\sigma=E\epsilon+\eta\dot{\epsilon}$	$\dot{\sigma}+\left(\frac{E_1+E_2}{\eta_2}\right)\sigma=E_1\dot{\epsilon}+\left(\frac{E_1E_2}{\eta_2}\right)\epsilon$	$\ddot{\sigma}+\left(\frac{E_M}{\eta_M}+\frac{E_M}{\eta_K}+\frac{E_K}{\eta_K}\right)\dot{\sigma}+\frac{E_ME_K}{\eta_M\eta_K}\sigma=E_M\ddot{\epsilon}+\frac{E_ME_K}{\eta_K}\dot{\epsilon}$
Constant stress sol'n (σ_0 applied at $t=0$)		$\epsilon=\sigma_0\left(\frac{1}{E}+\frac{t}{\eta}\right)$	$\epsilon=\frac{\sigma_0}{E}\left[1-\exp\left(-\frac{E}{\eta}t\right)\right]$	$\epsilon=\sigma_0\left\{\frac{1}{E_1}+\frac{1}{E_2}\left[1-\exp\left(-\frac{E_2}{\eta_2}t\right)\right]\right\}$	$\epsilon=\sigma_0\left\{\frac{1}{E_M}+\frac{t}{\eta_M}+\frac{1}{E_K}\left[1-\exp\left(-\frac{E_K}{\eta_K}t\right)\right]\right\}$
Constant strain rate sol'n ($\dot{\epsilon}=k_1$ applied at $t\geq 0$, $\sigma=0$)		$\sigma=k_1\eta\left[1-\exp\left(-\frac{E}{\eta}t\right)\right]$	$\sigma=k_1(\eta+Et)$	$\sigma=k_1\left\{\eta_2\left(\frac{E_1}{E_1+E_2}\right)^2\left[1-\exp\left(-\frac{E_1+E_2}{\eta_2}t\right)\right]+\frac{E_1E_2}{E_1+E_2}t\right\}$	$\sigma=k_1\left\{\frac{E_M}{r_1-r_2}(e^{r_1t}-e^{r_2t})+\frac{\eta_M}{r_1-r_2}(r_2e^{r_1t}-r_1e^{r_2t})+\eta_M\right\}$ where $r_1,r_2=-\frac{1}{2}\left(\frac{E_M}{\eta_M}+\frac{E_M}{\eta_K}+\frac{E_K}{\eta_K}\right)\pm\frac{1}{2}\left[\left(\frac{E_M}{\eta_M}+\frac{E_M}{\eta_K}+\frac{E_K}{\eta_K}\right)^2-4\left(\frac{E_ME_K}{\eta_M\eta_K}\right)\right]^{1/2}$
Constant stress rate sol'n ($\dot{\sigma}=k_2$ initiated at $t=0$, $\epsilon=0$)		$\epsilon=k_2\left(\frac{t}{E}+\frac{t^2}{2\eta}\right)$	$\epsilon=k_2\left\{\frac{t}{E}-\frac{\eta}{E^2}\left[1-\exp\left(-\frac{E}{\eta}t\right)\right]\right\}$	$\epsilon=k_2\left\{\left(\frac{1}{E_1}+\frac{1}{E_2}\right)t-\frac{\eta_2}{E_2^2}\left[1-\exp\left(-\frac{E_2}{\eta_2}t\right)\right]\right\}$	$\epsilon=k_2\left\{\left(\frac{1}{E_M}+\frac{1}{E_K}\right)t+\frac{1}{2\eta_M}t^2-\frac{\eta_K}{E_K^2}\left(1-\frac{\eta_K}{\eta_M}\right)\left[1-\exp\left(-\frac{E_K}{\eta_K}t\right)\right]\right\}$
Complex modulus $Y(i\omega)=Y_1+iY_2$	Y_1	$\frac{\omega^2\eta^2E}{E^2+\omega^2\eta^2}$	E	–	–
	Y_2	$\frac{\omega\eta E^2}{E^2+\omega^2\eta^2}$	$\omega\eta$	–	–
Complex compliance $J(i\omega)=J_1+iJ_2$	J_1	$\frac{1}{E}$	$\frac{E}{E^2+\omega^2\eta^2}$	$\frac{1}{E_1}+\frac{E_2}{E_2^2+\omega^2\eta_2^2}$	$\frac{1}{E_M}+\frac{E_K}{E_K^2+\omega^2\eta_K^2}$
	J_2	$\frac{1}{\omega\eta}$	$\frac{\omega\eta}{E^2+\omega^2\eta^2}$	$\frac{1}{\eta_2}-\frac{E_2^2}{\omega\eta_2(E_2^2+\omega^2\eta_2^2)}$	$\frac{1}{\omega\eta_M}+\frac{1}{\omega\eta_K}-\frac{E_K^2}{\omega\eta_K(E_K^2+\omega^2\eta_K^2)}$

FIGURE 1. Summary of basic rheological relationships.

For constant strain rate $\dot{\epsilon} = k_1$, applied at $t = 0$, $\sigma = 0$, the required solution is:

$$\sigma = k_1\left[\frac{E_M}{r_1 - r_2}(e^{r_1 t} - e^{r_2 t}) + \frac{\eta_M}{r_1 - r_2}(r_2 e^{r_1 t} - r_1 e^{r_2 t}) + \eta_M\right] \tag{3}$$

where

$$r_1, r_2 = -\frac{1}{2}\left(\frac{E_M}{\eta_M} + \frac{E_M}{\eta_K} + \frac{E_K}{\eta_K}\right) \pm \frac{1}{2}\left[\left(\frac{E_M}{\eta_M} + \frac{E_M}{\eta_K} + \frac{E_K}{\eta_K}\right)^2 - 4\left(\frac{E_M E_K}{\eta_M \eta_K}\right)\right]^{1/2}$$

In effect, equation (3) gives the stress/strain curve for the standard uniaxial loading at constant strain rate, since t can be replaced throughout by ϵ/k_1.

For constant stress rate $\dot{\sigma} = k_2$, initiated at $t = 0$, $\epsilon = 0$, the required solution is:

$$\epsilon = k_2\left\{\left(\frac{1}{E_M} + \frac{1}{E_K}\right)t + \frac{1}{2\eta_M}\cdot t^2 - \frac{\eta_K}{E_K^2}\left(1 - \frac{\eta_K}{\eta_M}\right)\left[1 - \exp\left(-\frac{E_K}{\eta_K}t\right)\right]\right\}. \tag{4}$$

The Burgers model is very convenient for qualitative description of deviatoric straining in the absence of bulk stress (pure shear), but its quantitative use is severely restricted by the inherent nonlinearity of ice viscosity and by the effective nonlinearity of both elasticity and viscosity which is caused by volumetric straining.

Three-element model

In elementary rheology a three-element model consisting of a spring in series with a Kelvin–Voigt unit is usually considered. The relevant equations are given in Fig. 1; they represent a degeneration of the equations for the Burgers model with $\eta_M = \infty$, although in the solution for constant strain rate the correspondence is not immediately obvious. An alternative three-element model can be obtained by retaining the Maxwell dashpot and eliminating the Maxwell spring.

Two-element model

The simple Maxwell and Kelvin–Voigt models give representations that are qualitatively incomplete for snow under typical loading conditions, but they may sometimes provide useful simplifications for analytical purposes. It is usual to assume a two-element model for analysis of vibrational tests designed to measure dynamic modulus, dynamic viscosity, loss tangent, relaxation time, etc. The Kelvin–Voigt is the usual choice.

Sinusoidal oscillation applications for models

With sinusoidal oscillation of stress or strain there is viscous energy dissipation and consequently a phase lag between stress and strain. For alternation at angular frequency ω with stress and strain amplitudes of σ_0 and ϵ_0, stress/strain relationships are expressed as

$$\sigma = Y(i\omega)\,\epsilon_0 e^{i\omega t} \tag{5}$$

$$\epsilon = J(i\omega)\,\sigma_0 e^{i\omega t} \tag{6}$$

where $Y(i\omega)$ and $J(i\omega)$, the complex modulus and complex compliance respectively, are defined as:

$$Y(i\omega) = Y_1 + iY_2 \tag{7}$$

$$J(i\omega) = J_1 + iJ_2 \tag{8}$$

Y_1 and J_1, the real parts, are the storage modulus and storage compliance respectively; these express elastic properties and give a measure of stored strain energy. The imaginary parts Y_2 and J_2 are the loss modulus and loss compliance respectively; they are related to energy dissipation by viscous damping, which is given by the loss factor, or loss tangent, $\tan\delta$, where δ is the lag angle between stress and strain:

$$\tan\delta = \frac{Y_2}{Y_1} = \frac{J_2}{J_1} \tag{9}$$

Expressions for Y_1, Y_2, J_1 and J_2 according to the various rheological models are given in Fig. 1. When elastic modulus (Young's modulus) and loss tangent have been measured, dynamic viscosity according to a certain model can be found from equation (9).

The properties discussed above are all frequency-dependent.

Simple elastic properties of coherent snow

Unless otherwise stated, the elastic modulus of snow is identified with the modulus of the Maxwell unit in the Burgers model. Data are commonly presented in the form of Young's modulus E, from which other elastic constants (bulk modulus K, shear modulus G, Lamé's constant λ) can be derived by the standard identities if Poisson's ratio ν is known:

$$K = \frac{E}{3(1-2\nu)} = \frac{2G(1+\nu)}{3(1-2\nu)} = \frac{\lambda(1+\nu)}{3\nu} = \frac{3\lambda+2G}{3} \tag{10}$$

$$G = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3(K-\lambda)}{2} \tag{11}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2G\nu}{(1-2\nu)} = \frac{3K\nu}{(1+\nu)} = \frac{3K-2G}{3} \tag{12}$$

The most consistent data for dense snow are derived from field measurements of seismic velocity and laboratory measurements of pulse velocity or high frequency (10^2–10^3 Hz) flexural vibrations. Figure 2 gives a general impression of E as a function of snow density for cold snow, together with a summary of corresponding values of Poisson's ratio, as determined by dynamic methods.*

Traditional quasistatic uniaxial loading tests are inherently deceptive when applied to snow, since they produce stress/strain curves that represent the response of all elements of the Burgers model. If the test is run at high strain rates, the initial tangent modulus from a uniaxial stress/strain curve should give a good approximation to the 'Maxwell spring' modulus, provided that good experimental technique is employed. However, with low compliance specimens (dense and cold), contact imperfections at the loading platens will depress the apparent value of initial tangent modulus if displacement is measured across the platens rather than across the specimen itself. Figure 2 gives values of initial tangent modulus derived from careful tests on cold snow. The results for high density snow are lower than the dynamic values by a factor of 2; since the tests were run at high strain rates on 'stiff' snow, the discrepancy may well be attributable to the fact that strain was calculated from platen displacement. Extrapolation of the dynamic data suggests that uniaxial values for low density snow may be low by a factor of 5, probably because of the low strain rates and low viscosity of the snow.

The true effect of temperature on E is hard to determine. In uniaxial tests where apparent initial tangent modulus is rate-dependent, a change of temperature will have the effect of changing the rate or frequency by virtue of its strong influence on viscosity. The best estimates of true temperature effect are obtained from high

* In this and subsequent figures where a mechanical property is related to a single independent variable, the width of data bands partly reflects variation of secondary parameters (e.g. structural characteristics or temperature conditions).

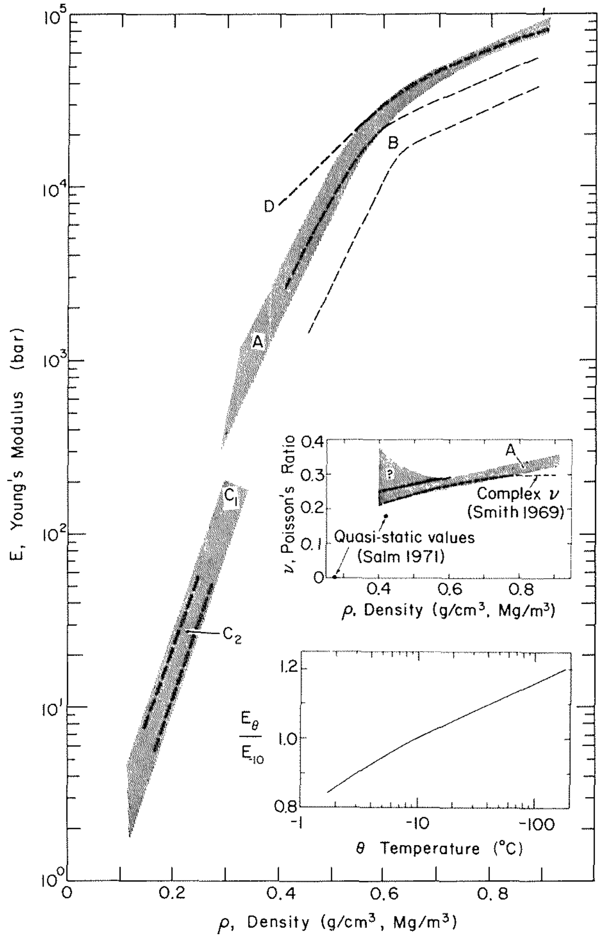


FIGURE 2. Young's modulus for dry, coherent snow. (A) Pulse propagation or flexural vibration at high frequencies, $-10°$ to $-25°C$ (Smith, 1965; Nakaya, 1959a, b; Bentley *et al.*, 1957; Crary *et al.*, 1962; Lee, 1961; Ramseier, 1963). (B) Uniaxial compression, strain rate approximately 3×10^{-3} to $2 \times 10^{-2} s^{-1}$, temperature $-25°C$ (Kovacs *et al.*, 1969). (C_1) Uniaxial compression and tension, strain rate approximately 8×10^{-6} to $4 \times 10^{-4} s^{-1}$, temperature $-12°$ to $-25°C$. (C_2) Static creep test, $-6.5°$ to $-19°C$ (Kojima, 1954). (D) Complex modulus, 10^3 Hz, $-14°C$ (N. Smith, 1969).

frequency vibration experiments, which suggest that E does not vary much over typical temperature ranges for dry snow (Fig. 2).

The data summarized in Fig. 2 refer to dry snow that is well bonded. Values for granular snow with low cohesion are smaller, and E increases exponentially with time during sintering, finally tending asymptotically to a limit. When dense snow is artificially disaggregated and then allowed to sinter at approximately $-10°C$, the initial value of dynamic modulus is typically about 40 per cent of the value reached after 3 weeks, and about 30 per cent of the value reached after 1 year or more.

Perhaps the most important conclusions that could be drawn for the theoretical mechanician concern the linearity of elasticity in snow. If the problem involves bulk stresses that are insignificant (e.g. pure shear or, more practically, small bulk stress in dense snow), then it may be justifiable to assume that snow is linearly elastic, permitting application of simple elastic theory. However, if bulk stresses are significant, density will increase irreversibly in response to increases of bulk stress, and the elastic modulus is a very strong function of density, varying by three orders of magnitude or more within the range of densities commonly encountered in field situations.

Viscoelastic properties of coherent snow

At first sight it appears quite feasible to determine separately the moduli and viscosity coefficients of the Burgers model from quasistatic tests, e.g. by analysing constant-stress creep curves according to equation (2). However, there are practical difficulties in making tests that do not involve significant bulk stress, and since elastic and viscous properties are highly sensitive to density change, simple tests such as the uniaxial compression test can be misleading. One alternative is to use vibrational techniques, such as have been described by Nakaya (1959a, b), Lee (1963), N. Smith (1969), and Chae (1967). A snow specimen is set in flexural, torsional or longitudinal oscillation, and viscoelastic properties are calculated from test measurements, usually by assuming either a Kelvin–Voigt or a Maxwell model for analysis (see equations (5)–(9) and Fig. 1).

Values for the real part of the complex modulus have already been discussed and given (Fig. 1); a plot of the complex modulus as a function of density has been added to Fig. 2 for comparison. Variation of the complex modulus with frequency is indicated in Fig. 3.

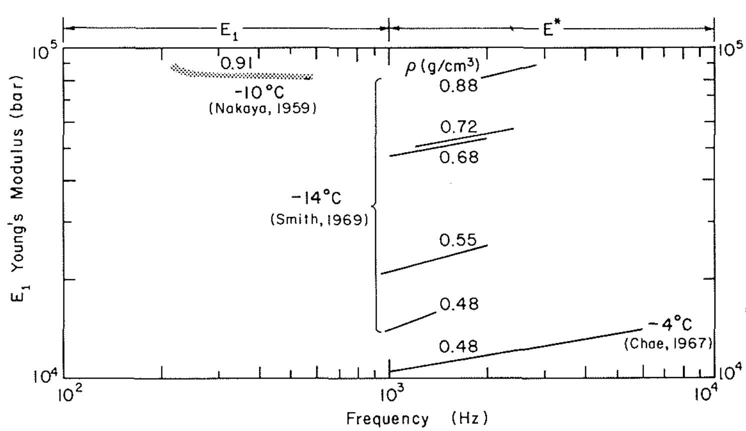


FIGURE 3. Dynamic Young's modulus as a function of frequency.

Variation of the loss factor $\tan\delta$ with frequency does not appear to be very well defined, although there is clearly a decrease with increasing frequency (Fig. 4). The available data for loss tangent as a function of density (Fig. 5) are confined mainly to densities higher than those normally found in seasonal snowpacks, but there are indications that $\tan\delta$ rises to high values, say 0.4, at densities representative of surface snow layers. Nakaya's (1959a, b) data indicate that $\tan\delta$ is a very strong function of temperature between -5° and -20°C, although rough interpolation from other data suggests a less strong temperature-dependence in this range (Fig. 6). Between -50° and -90°C, maxima in $\tan\delta$ occur; these are reminiscent of similar features that relate to dielectric processes.

Dynamic viscosity η can be calculated from the loss factor when a suitable rheological model is assumed. For a Maxwell model, $\eta = E/(\omega \tan\delta)$, and for a Kelvin–Voigt model, $\eta = (E \tan\delta)/\omega$. In Fig. 7 dynamic viscosity values are plotted against frequency, indicating a decrease with increasing frequency. Dynamic viscosity is plotted against density in Fig. 8, showing a rapid increase of viscosity with increasing

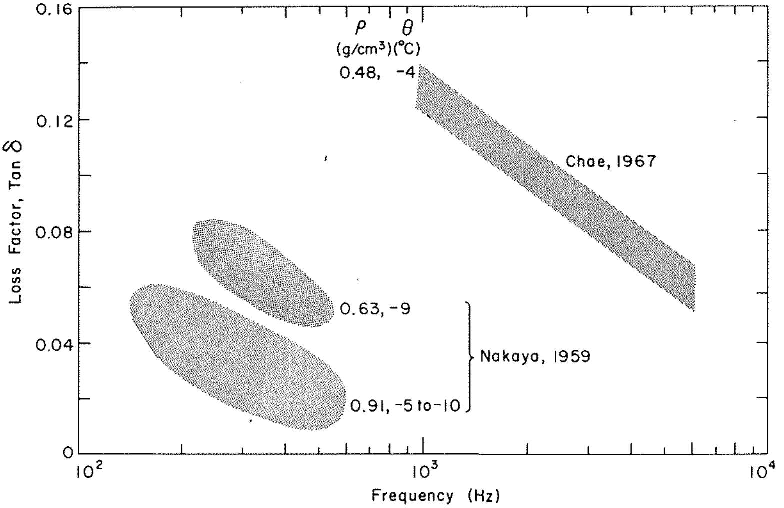


FIGURE 4. Loss factor plotted against frequency.

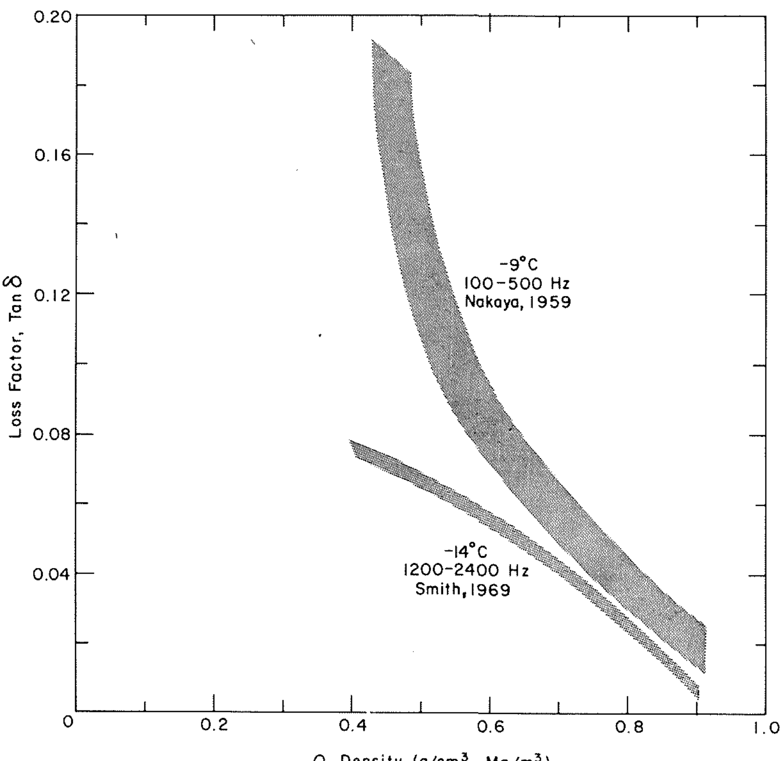


FIGURE 5. Loss factor as a function of density.

density. Variation of dynamic viscosity with temperature does not appear to have been investigated systematically for snow, but Nakaya's (1959a) results for ice indicate variation according to an Arrhenius-type relation with activation energy values ranging from 12.7 to 18.7 kcal/mole.

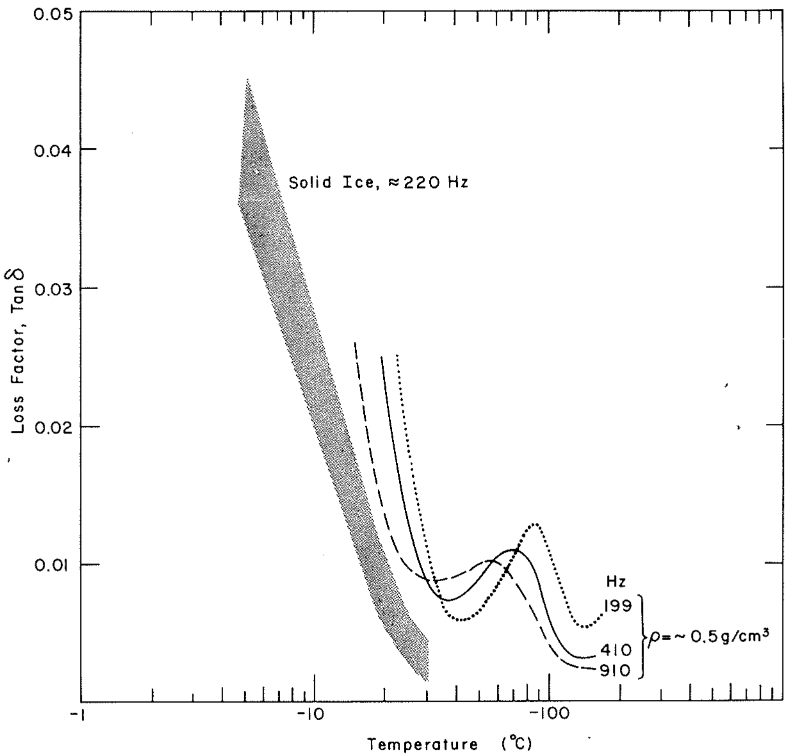


FIGURE 6. Loss factor as a function of temperature (data from Nakaya, 1959a, b).

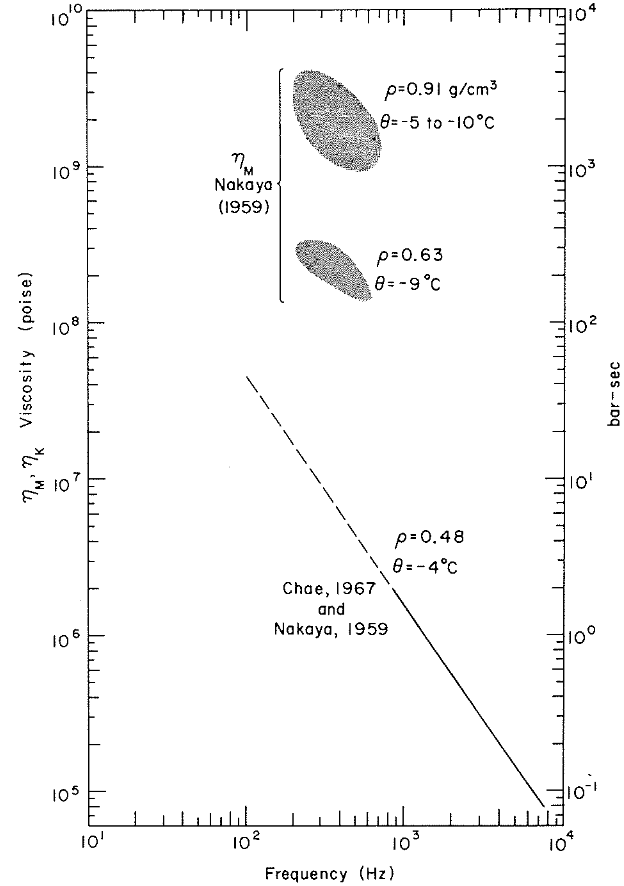


FIGURE 7. Dynamic viscosity as a function of frequency.

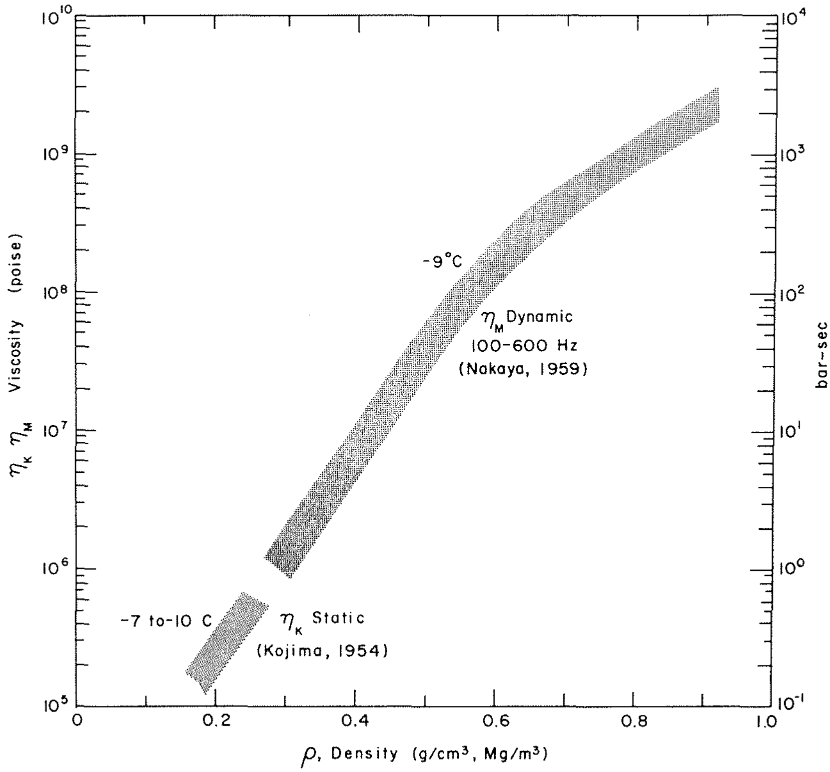


FIGURE 8. Viscosity as a function of density.

Practical experimental difficulties have inhibited application of dynamic methods to low density snow, so that data are lacking in this region. Kojima (1954) undertook quasistatic creep tests on low density snow in order to determine the moduli and viscosities of the Burgers model, and his values of η_M have been plotted in Fig. 8.

One possible source of trouble in dynamic measurements at high temperatures might be temperature rise in the specimen due to internal energy dissipation.

Relaxation times for high-frequency vibration (given by η_K/E_K, or $\tan \delta/\omega$) appear to be in the range 10^{-5} to 10^{-3} s, which is suggestive of molecular processes within the individual ice crystals (dielectric relaxation times are of the same order). However, relaxation times derived from quasistatic measurements (Kojima, 1954; Shinojima, 1967) are very much greater (5–2000 s),* and Shinojima (1967) finds τ decreasing with strain rate (approximate inverse proportionality) for the range 8×10^{-6} to $4 \times 10^{-4}\,s^{-1}$. The latter suggests a bulk property that may be dependent on structural arrangement and connection of constituent crystals.

Viscous behaviour in sustained creep

Relaxation times for the Kelvin–Voigt unit are short, and so for sustained creep processes the stress/strain/time relationships ought to be controlled by the Maxwell viscosity η_M. The most direct way to study this property is to make constant-stress creep tests and thus obtain relationships between stress and strain rate for secondary creep. However, such results tend to be complicated by volumetric straining if bulk stress is significant; this complication can be avoided by employing a pure shear test (torsion of a hollow cylinder; application of equal-magnitude, opposite-sign principal stresses), or by applying stresses that are small relative to the creep resistance (density).

* Ramseier and Pavlak (1964) report values up to 8×10^6 s.

Unfortunately, adequate precautions have been lacking in some test programmes, and the resulting stress/strain-rate data could be somewhat misleading. Nevertheless, creep test data show quite clearly that for snow of a given density the stress/strain-rate relation for secondary creep is strongly nonlinear. The latter fact has an important bearing on the interpretation of viscosity data obtained by less direct means.

'Uniaxial stress viscosity'

Very simple techniques have often been used for making creep tests under uniaxial compressive stress. For example, a cylinder or prism of snow is subjected to constant axial deadload and allowed to deform until a plot of displacement against time *appears* to become linear, at which time final strain rate is recorded against initial stress and initial density. This method works well as long as the stress level is low and total strains are small, but otherwise it is unsatisfactory unless special data analysis methods are adopted. It can be particularly misleading when stress/strain-rate relationships are being investigated.

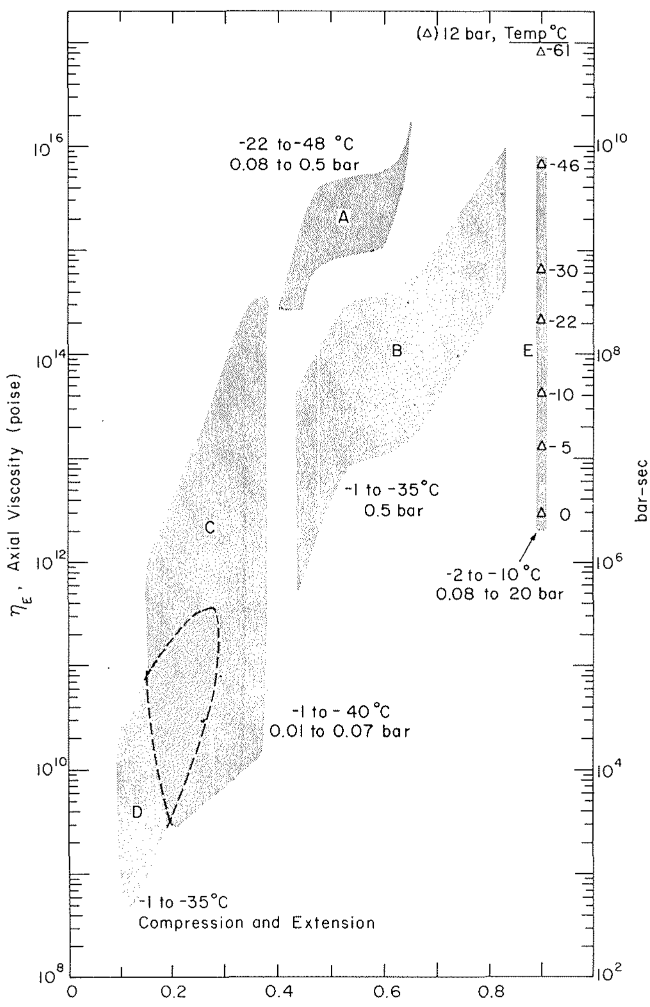


FIGURE 9. 'Axial viscosity' (principal stress/strain-rate ratio for creep under uniaxial stress) plotted against density. (A) Ramseier and Pavlak, 1964. (B) Mellor and Smith, 1967. (C) Bucher, 1948. (D) Shinojima, 1967. (E) Mellor and Smith, 1967; Mellor and Testa, 1969.

Uniaxial test results are not normally analysed to separate dilatational and deviatoric effects, and data are usually in the form of axial stress/strain-rate relationships, or in terms of an 'axial viscosity'. This 'uniaxial stress viscosity' is the ratio of axial stress to axial secondary strain rate; it is analogous to Young's modulus, and here is designated η_E. Figure 9 gives a general impression of the magnitude of η_E, and it is immediately evident that values fluctuate wildly. For a given material, η_E can vary by 4 to 5 orders of magnitude as temperature and stress level vary within conceivable practical ranges.

Variation of creep viscosity with temperature is usually represented by an Arrhenius relation (see Fig. 10). Values of the appropriate activation energy found by direct experiment lie in the range 11–18 kcal/mole (Mellor and Smith, 1966), while other test data (Bucher, 1948) suggest values from about 9–20 kcal/mole. More careful experiments give a value of 16.4 kcal/mole for polycrystalline ice, and show quite clearly that the Arrhenius relation breaks down at temperatures above −10°C with ice of typical chemical purity (Mellor and Testa, 1969). Although it has a basis in chemical rate theory, the Arrhenius equation gives a poor mathematical description of temperature-dependence for the viscosity of snow (Mellor and Smith, 1966) and for some other properties that are controlled by more than one activated process or mechanism. Alternative empirical equations (e.g. simple exponential) could be considered for application to constitutive equations.

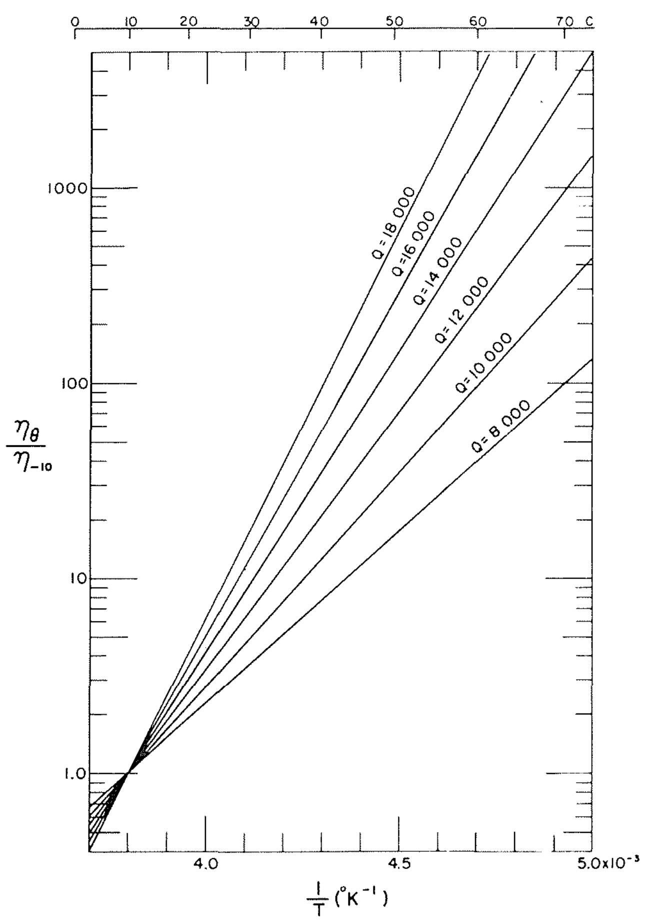


FIGURE 10. Variation of viscosity with temperature according to Arrhenius relationship.

'Compactive viscosity'

About 20 years ago, Bader (1953) drew attention to Sorge's (1935, 1938) observations and deductions relating to the time-invariance of Greenland snow density profiles; he went on to formalize steady-state relationships between density ρ and accumulation rate $\dot{A}$ (Bader, 1960, 1962), and defined a 'compactive viscosity' η_c as the ratio of overburden pressure σ_v to vertical strain rate $\dot{\epsilon}_v$ at any given depth z in a flat-lying deposit of dry snow:

$$\eta_c = \frac{\sigma_v}{\dot{\epsilon}_v} \tag{13}$$

$$\sigma_v = \int_0^z \rho g \, dz = \dot{A} t_s \tag{14}$$

$$V = \frac{dz}{dt} = \frac{dz}{d\sigma} \cdot \frac{d\sigma}{dt} = \frac{\dot{A}}{\rho g} \tag{15}$$

$$\dot{\epsilon}_v = \frac{dV}{dz} = -\frac{\dot{A}}{\rho^2 g} \cdot \frac{d\rho}{dz} = -\frac{1}{\rho} \frac{d\rho}{dt} \tag{16}$$

where t_s is the elapsed time since the layer in question was deposited at the surface and V is the velocity at which it sinks below the surface. Viscosity values obtained from this ingenious scheme were used with considerable success in engineering analyses of foundation settlement and tunnel closure in Greenland and Antarctica. The concept was also applied to settlement of seasonal snowpacks by other investigators.

Compactive viscosity has been widely regarded as characterizing continuous steady creep, Bader himself assuming newtonian viscosity for stresses less than 0.8 bar, but in view of the demonstrated nonlinearity of viscosity and the implicit relation between σ_v and ρ in determinations of η_c, a careful appraisal seems called for.

Sequential measurements of density in snow deposits yield diagrams known as 'layer profiles', in which the density of a given layer is plotted against time, and increments of new snow accumulation are recorded. Layer profiles for seasonal snowpacks show that when new snow is added to the surface, each underlying layer densifies rapidly for a few days, but gradually the vertical creep decays to a relatively insignificant rate. The same behaviour is found in uniaxial-strain compressive creep tests. Thus densification occurs largely by a quasiplastic collapse of limited duration rather than by a steady creep, and apparent values of η_c are partly dependent on the interval between snowfalls.

This type of densification is in the nature of an equilibration process. After each event the stress is at a level which produces only very slow creep, but any significant addition to that stress gives disproportionately rapid creep, a situation similar to the equilibration of maximum deviator stress in glaciers. Thus η_c, which must be nonlinear with respect to stress or strain rate, is automatically determined for a stress level that is critical for the prevailing snow density.

Figure 11 gives values of η_c calculated from field observations, together with some results from uniaxial-strain creep tests. The discrepancy between values for polar snowfields and for seasonal snowfields seems too great to be explained by temperature difference alone; it is possible that the seasonal snow values reflect basic shortcomings of the concept when large discrete snowfalls occur at frequent intervals. In view of the uncertainties surrounding the determination and interpretation of η_c, it may be safer to work directly with stress/strain-rate data.

Sustained creep is of overwhelming importance in snow mechanics, but in view of the complications arising from nonlinearity of viscosity with stress and density, it is necessary to have some systematic framework for analysis of viscous deformation in multiaxial stress fields.

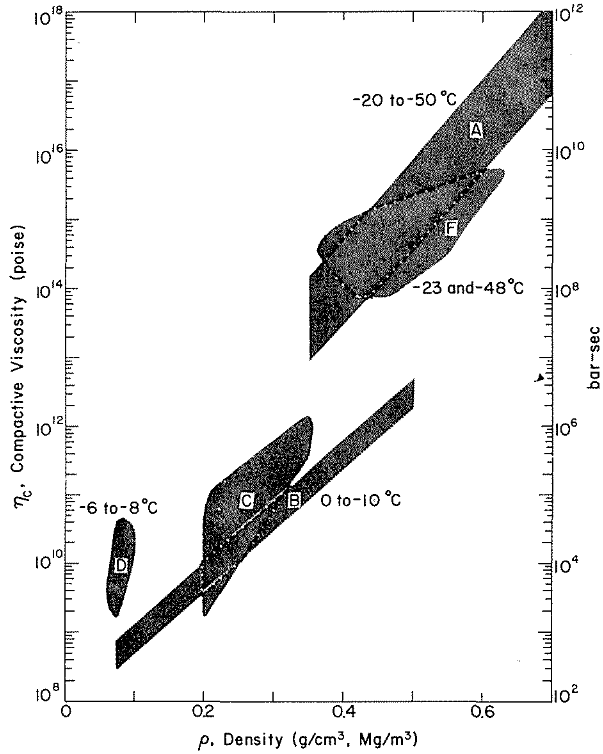


FIGURE 11. 'Compactive viscosity' (major principal stress/strain-rate ratio for creep under uniaxial strain) plotted against density. (A) Greenland and Antarctica, −20° to −50°C (Bader, 1963). (B) Seasonal snow, Japan, 0° to −10°C (Kojima, 1967). (C) Alps and Rocky Mts (Keeler, 1969). (D) Uniaxial-strain creep tests, −6° to −8°C (Keeler, 1969). (F) Uniaxial-strain creep tests, −23° and −48°C (Mellor and Hendrickson, 1965).

Simple stress analysis

Further discussion of mechanical behaviour is facilitated by separating volumetric, or dilatational, effects from pure shear, or deviatoric, effects. The general stress tensor σ_{ij} can be split into deviatoric components σ'_{ij}, which tend to change the shape of an element without change of volume, and a dilatational (isotropic) component, or bulk stress $\bar{\sigma}$, which tends to change only specific volume:

$$\sigma_{ij} = \sigma'_{ij} + \delta_{ij}\bar{\sigma} = \sigma'_{ij} + \tfrac{1}{3}\delta_{ij}\sigma_{kk} \tag{17}$$

($i, j, k = 1, 2, 3$, or x, y, z; $\delta_{ij} = 1$ when $i = j$, $\delta_{ij} = 0$ when $i \neq j$; repeated suffix denotes summation).

It is also convenient to express stresses in such a way that they are independent of the directions of coordinate axes. Stress invariants I_1, I_2, I_3 are expressed in terms of cartesian components σ_{xx}, σ_{xy}, etc., or principal stresses $\sigma_1, \sigma_2, \sigma_3$, as:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 \tag{18}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{19}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} = \sigma_1\sigma_2\sigma_3 \tag{20}$$

The bulk stress, or octahedral normal stress, $\bar{\sigma}$ is

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3}I_1 \tag{21}$$

The octahedral shear stress τ_{oct}, which is also independent of coordinate axes, is

$$\tau_{oct} = \tfrac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} = \sqrt{(2)}/3\,(I_1^2 + 3I_2)^{1/2} \tag{22}$$

Corresponding relationships exist for strain ϵ_{ij}, and for strain rate $\dot{\epsilon}_{ij}$.

A 'bulk viscosity' $\bar{\eta}$, analogous to the elastic bulk modulus K, and a 'shear viscosity' μ, analogous to the elastic shear modulus G, can be defined such that

$$\tfrac{1}{3}\sigma_{ii} = \bar{\eta}\dot{\epsilon}_{ii} \tag{23}$$

$$\sigma'_{ij} = 2\mu\dot{\epsilon}'_{ij} \tag{24}$$

and therefore, if isotropy is assumed:

$$\sigma_{ij} = 2\mu\dot{\epsilon}'_{ij} + \delta_{ij}\bar{\eta}\dot{\epsilon}_{kk} \tag{25}$$

Under this scheme, test data can be analysed to yield constants that are potentially applicable to general multiaxial stress states. However, there are still serious problems to be solved, since $\bar{\eta}$ and μ are nonlinear with respect to stress, and they vary greatly with density (or specific volume), strain rate, and temperature. Salm (1967) has approached these problems by assuming $\bar{\eta}$ and μ to be functions of the invariants of strain rate, but other approaches can be taken. This is the essence of the problem in deriving a viscous, or viscoplastic, constitutive equation for snow.

From equations (23) and (24), compactive viscosity η_c and 'uniaxial stress viscosity' η_E can be expressed in terms of $\bar{\eta}$ and μ:

$$\eta_c = \bar{\eta} + \tfrac{4}{3}\mu \tag{26}$$

$$\eta_E = \frac{9\bar{\eta}\mu}{3\bar{\eta} + \mu} \tag{27}$$

With a matched pair of tests giving η_c and η_E for the same snow under similar conditions, values of $\bar{\eta}$ and μ can be determined from equations (26) and (27). However, data sets of this kind are not normally available, and it may be necessary to resort to more devious methods, such as estimation of the viscous analogue of Poisson's ratio ν_v, which may be defined as

$$\nu_v = \frac{3\bar{\eta} - 2\mu}{2(3\bar{\eta} + \mu)} \tag{28}$$

Figure 12(a) gives a summary of experimental data for ν_v. These results are widely scattered, but a suggested envelope of probable values for compression is given in Fig. 12(b).

For most quasistatic problems in snow mechanics, gravity body forces are significant or even dominant, and they have to be incorporated into the equilibrium equations, which for negligible acceleration are:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0 \tag{29}$$

The equation representing resolution normal to the free surface of an extended snow mass can usually be integrated directly to give one component of normal stress (which may or may not be a principal stress), as in the case of densification theory. Other components of normal stress can sometimes be estimated with acceptable accuracy by assuming an appropriate value of ν_v.

Volumetric strains, irreversible compressibility, and equations of state

With typical engineering materials (e.g. metals, solid plastics, dense ceramics, rock, concrete) there is usually an obvious reference state for the unloaded material;

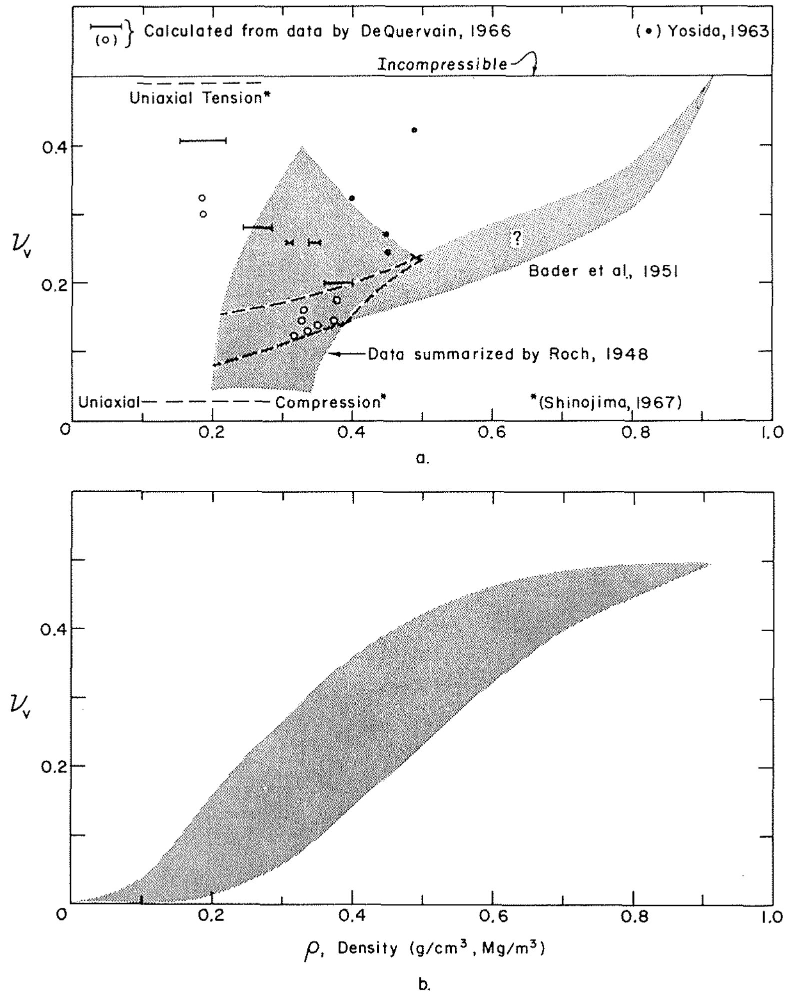


FIGURE 12. Summary of data for viscous analogue of Poisson's ratio (a), and speculative envelope for viscous analogue of Poisson's ratio (b).

volumetric strains tend to be small and substantially reversible up to the point of incipient failure of the material, and it is quite realistic to describe the volumetric stress/strain behaviour by a simple rate-dependent bulk modulus.

Typical snow is very different, in that compressive volumetric strains are large and substantially irreversible, resistance to deviatoric straining is highly sensitive to volumetric strain, and in many problems it is unrealistic to interpret irreversible volumetric strain as 'failure'. Thus it is not sufficient to characterize the mechanical behaviour of snow solely in terms of the original physical state (density, etc.) and the state of stress; it is necessary to also account for the stress or strain history.

In principle, the last statement applies to virtually all materials, and in rational mechanics there exist formal procedures that account for stress or strain history. However, the practical difficulties of treating stress history are formidable, and with snow the difficulties are accentuated by the time-dependent structural changes that can occur almost independently of stress (sintering, grain growth, etc.).

In those branches of solid mechanics where compressibility and large volumetric strains are major considerations, it is usual to define 'equations of state', which give relationships between hydrostatic pressure and specific volume. Since temperature and strain rate are controlling parameters, there are innumerable pressure/volume relations for any given material; of special interest are the relations for slow isothermal compression and for adiabatic shock compression. For materials with finite yield stress, the 'isotherm' in a pressure/volume plot sets a lower limit to the family of compressibility curves, while the shock adiabat sets an upper limit. The latter relation, which represents the locus of final states for shock compression, is known as a Hugoniot, or sometimes as the Rankine–Hugoniot equation of state; the actual stress/strain path between the initial state and any given final state is known as the Rayleigh line.

When snow is compressed hydrostatically, density increases quite readily until the grains are 'close-packed', after which the snow becomes more resistant, and there is more gradual compression to porous but impermeable ice, which in turn can be compressed to solid bubble-free ice. If compression is continued, the Ice I-h formed from snow undergoes successive transformations to high-pressure polymorphs. Since density is the reciprocal of specific volume in dimensional form, any graph that gives a systematic relationship between pressure and density can be interpreted as an equation of state. As already mentioned, the upper limit to the system of pressure/density curves is set by the high-rate shock adiabat, but because snow does not appear to have a finite yield stress there is, in principle, no lower limit to the curves representing isothermal compression. However, for practical purposes there does appear to be a lower limit; it represents the maximum density that can be reached under a given sustained pressure in time periods of the order of months or more. This has already been referred to in the discussion of compactive viscosity.

True hydrostatic tests are rarely made, and compressibility characteristics have to be deduced mainly from uniaxial strain experiments, results of which are summarized in Fig. 13. The same data are replotted in terms of bulk stress $\bar{\sigma}$, or first stress invariant I_1, in Fig. 14, the transformation being effected by assuming appropriate values of ν_v (see Fig. 12) and calculating secondary principal stresses for the uniaxial strain conditions.

In attempting to define the adiabatic upper limit to the compressibility characteristics, only one set of Hugoniot data (Napadensky, 1964) can be drawn upon; the only other known attempt to measure these characteristics (J. Smith, 1969) involved shock tube experiments that produced volumetric strains far too small to yield meaningful Hugoniots. The original results by Napadensky clearly contain gross errors (densities in excess of 1.1 g/cm^3 at pressures of the order of 100 bar), but for the range plotted in Figs. 13 and 14 the data are probably reasonably valid. No Hugoniot data are available for snow of typical densities, but an effort has been made to fill the gap by drawing on calculations for high-speed impact stress (Mellor, 1968). These are based on mass and momentum conservation, and assumption of a limiting density for compaction under plane-wave impact. Kinosita's (1967) compression data, which were generated at rather low rates with a somewhat ill-defined stress state, converge with the Napadensky Hugoniot data, and this suggests that it may be possible to obtain a good approximation to the high-rate compressibility characteristics by simple uniaxial strain compression tests on a modern testing machine (the high-speed actuator of CRREL's closed-loop servo machine has a maximum controllable speed of 0.42 m/s, which is almost 4 orders of magnitude higher than the speeds used in Kinosita's experiments). However, for impact at very high speeds it seems likely that there could be stresses significantly higher than any shown in Figs. 13 and 14. Dr G. K. Swinzow of CRREL finds that when inert projectiles of steel and aluminium are fired at velocities around 10^3 m/s into snow of density 0.4–0.5 g/cm^3, the metals (which have yield stresses of the order of 3×10^3 to 5×10^3 bar) are deformed plastically, suggesting stresses of the order found in Hugoniots for ice. This condition

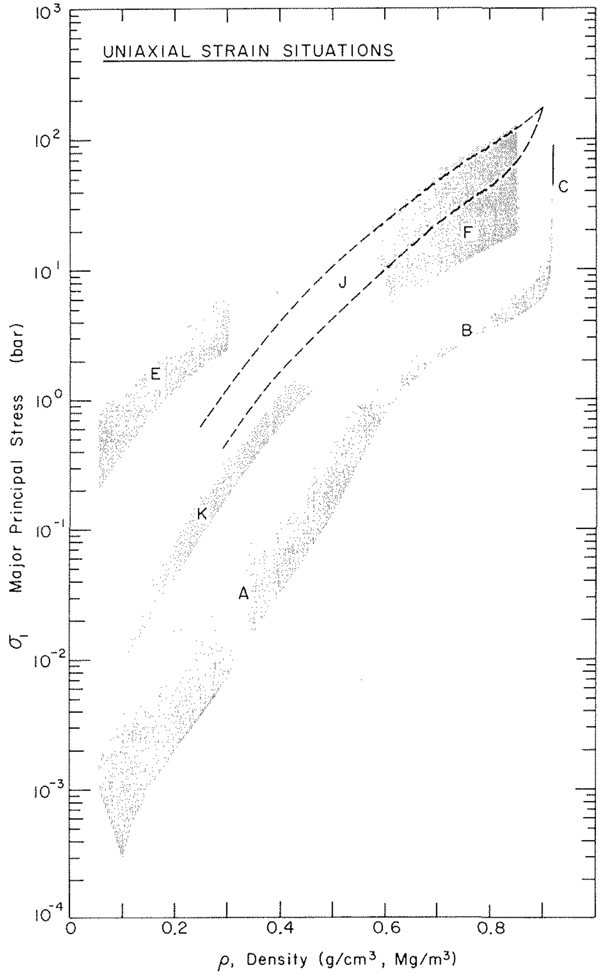


FIGURE 13. Compilation of data relating major principal stress to bulk density for compression in uniaxial strain at various rates and temperatures. (A) Natural densification of snow deposits, $-1°$ to $-48°C$. Approximate average loading rates 10^{-10} to 10^{-8} bar/s (data from depth/density curves for many sites). (B) Slow natural compression of dense firn and porous ice. Approximate loading rates 10^{-10} to 10^{-9} bar/s (from depth/density curves for polar ice caps). (C) Slow compression of solid ice. (D) Isothermal compression of high pressure polymorphs of ice, $-10°C$ (Bridgman, 1911, 1937). (E) Calculated values for plane wave impact at 20–40 m/s (Mellor, 1968). (F) Hugoniot data for explosively generated shock waves (impact velocities 1–12 m/s), $-7°$ to $-18°C$ (data selected from Napadensky, 1964). (G) Hydrostatic compression under constant stress at $-10°C$. Strain rates 10^{-6} to $10^{-3}\,s^{-1}$ (Landauer, 1957). (H) Hugoniot data for solid ice at $-10°C$ (Anderson, 1968). (I) Hugoniot data for solid ice at $-10°C$ (Nakano and Froula, 1973; Larson *et al.*, 1973). (J) Compression at approximately constant strain rate, $-7°$ to $-18°C$. Strain rate $\approx 10^{-4}\,s^{-1}$ (Kinosita, 1967). (K) Compression in uniaxial strain – incremental loading to collapse, $-2°$ to $-3°C$ (Bucher and Roch, 1946).

of virtual incompressibility is probably reached when the velocity of the plastic 'wave' (density jump) approaches the elastic wave velocity of the snow.

Compressibility curves for slow isothermal straining can be obtained from depth–density relations. Integration of unit weight with respect to depth gives a relation between major principal stress and density for uniaxial strain, and a conversion to bulk stress can be made with the aid of estimated values of ν_V, as mentioned above.

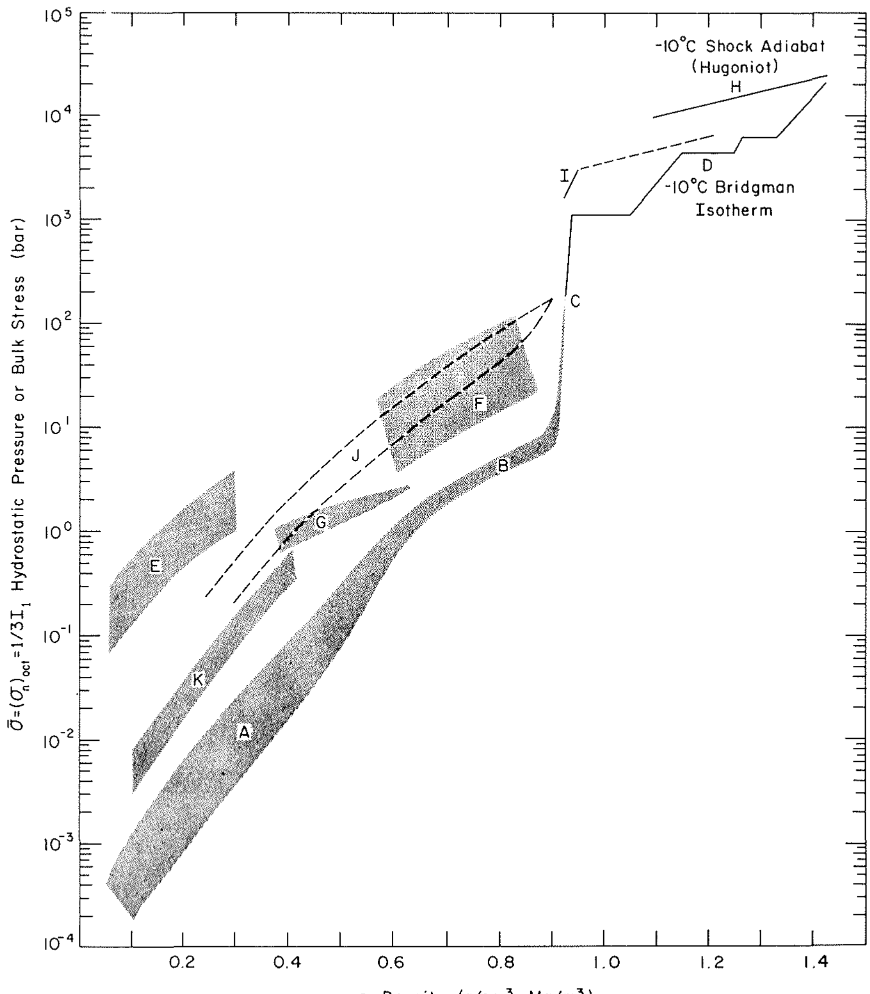


FIGURE 14. 'Equation of state' data deduced from Fig. 13, together with equation of state data for zero-porosity ice; density is the reciprocal of dimensional specific volume (see legend for Fig. 13).

Figures 13 and 14 give a data band obtained in this way; dispersions due to variation in initial surface density and to meltwater complications have been disregarded. This band, which appears to set a practical lower limit to the compressibility domain, represents stress rates from about 10^{-10} to 10^{-8} bar/s, and temperatures in the range $-1°$ to $-48°C$. In the isothermal range, the compressibility curve would be expected to displace upward with increasing stress rate and decreasing temperature.

With systematically derived compressibility curves (control of temperature and rate parameters), intersections with an ordinate for constant density would give the rate-dependence of volumetric stress/strain relations. Currently available data are inadequate for this purpose, but for medium-density snow it appears that the compressibility domain spans 3 decades of stress for a corresponding span of about 12 decades of stress rate or strain rate, which suggests that strain rate may be proportional to the fourth power of stress in this range.

In the absence of reliable information on rate-dependence, some problems, including those involving low stress rates or strain rates, can perhaps be tackled on the assumption that density equilibrates with compressive bulk stress by a quasiplastic collapse process, disregarding the fact that finite time is needed for equilibration. If equilibrium stress/density data are available for a specified range of stress rates and temperatures, e.g. the lowest band in Fig. 14, they can be expressed in terms of a compression modulus K_p that is a plastic analogue of the elastic bulk modulus:

$$K_p = \rho \frac{d\sigma}{d\rho} \tag{30}$$

Figure 15 gives a plot of K_p for density equilibrations in snowpacks, and compares values with those for elastic bulk modulus K.

Deviatoric stress/strain-rate relations at constant specific volume

In attempting to separate dilatational and deviatoric effects, it is helpful to consider situations in which dilatational effects can be neglected. Such situations include pure

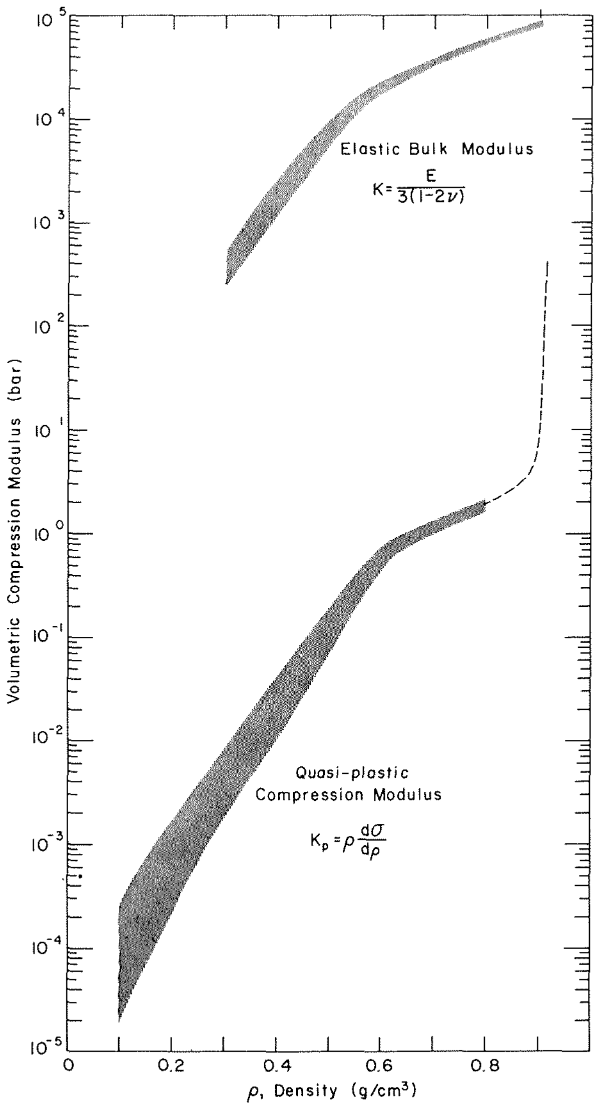


FIGURE 15. Volumetric compression moduli plotted against density.

shear stress states, and conditions in which bulk stress is below the limits set by the appropriate compressibility curve. The objective here is to deduce deviatoric stress/strain-rate relations for snow of constant density, with special emphasis on the relation for secondary creep.

Constant stress creep tests might appear to be an obvious source of data, but actually only those tests in which the bulk stress is below the lowest limit of Fig. 14 are directly useful. This tends to rule out most of the tests made for engineering purposes, and data have to be sought from less obvious sources.

Tests designed to measure uniaxial compressive strength apply an approximately constant strain rate until the specimen fails. In a test where the duration exceeds the relaxation time of the specimen, the stress–strain diagram curves through a maximum at failure, yielding a maximum ratio of stress to strain rate that corresponds to the value obtained at the secondary creep inflection point of a constant stress creep test. If it can be shown that this type of test produces failure without excessive volumetric strain, then its results give a limiting value for the stress/strain-rate relation of the snow under consideration. In view of the complications described in the previous section it is hard to decide whether typical test durations exceed, or are comparable to, the appropriate relaxation times, but stress/strain curves are certainly continuous through a maximum when careful techniques are employed. With dense, dry snow, failure in uniaxial compression does not appear to be preceded by volumetric collapse in typical tests; apparent values of Poisson's ratio do not suggest large dilatations during loading, and the bulk stress at failure is below the compressibility curve for comparable loading rates. Thus it will be assumed here that approximate limiting values for deviatoric stress/strain-rate relations can be deduced from results of uniaxial strength tests (Fig. 16).

Another unconventional data source is provided by deformation observations made on uniform snow slopes (Mellor, 1973). At the end of the net accumulation period, seasonal snowpacks have reached a stress/density equilibrium; volumetric straining has virtually ceased, but on slopes the snow continues to shear under gravity body forces. Stresses are easily calculated for long uniform slopes, and shear strain rates can be measured by observing the deflection of compliant columns set into the snow. It is rather remarkable that under these conditions the shear strain rate is practically invariant with depth (and therefore invariant with shear stress). The logical conclusion seems to be that shear resistance, being proportional to shear stress, is therefore proportional to overburden pressure or bulk stress (shear stress and normal pressure are proportional for constant slope angle). This is a very important clue to the relationship between volumetric and deviatoric straining.

Observations made in the Alps and Rocky Mountains indicate that snow slopes of approximately 30° inclination have depth invariant deviatoric strain rates in the range 1.1×10^{-8} to $2.8 \times 10^{-8}\ s^{-1}$ (Haefeli, 1939; Martinelli, 1960; Frutiger and Martinelli, 1966; Judson, 1973). If a relationship between normal stress and density is computed from corresponding depth–density profiles or, alternatively, taken from data compilations such as Fig. 13, then shear stress can be calculated for any given density, since the ratio between shear stress resolved downslope and the stress normal to the slope plane is equal to the tangent of the slope angle. Data thus obtained from 30° slopes are plotted in Fig. 16.

Figure 16 attempts to give an impression of relations between deviatoric components of stress and strain rate, with density as parameter, for conditions where volumetric straining is insignificant. The relationship for low porosity ice, which is also shown in Fig. 16, is well established, and this lends a certain plausibility to the trends that are indicated for snow on the basis of very scanty data. It would be well worthwhile to supplement the data of Fig. 16 by means of low stress creep tests and observations on natural slopes of widely varying inclination.

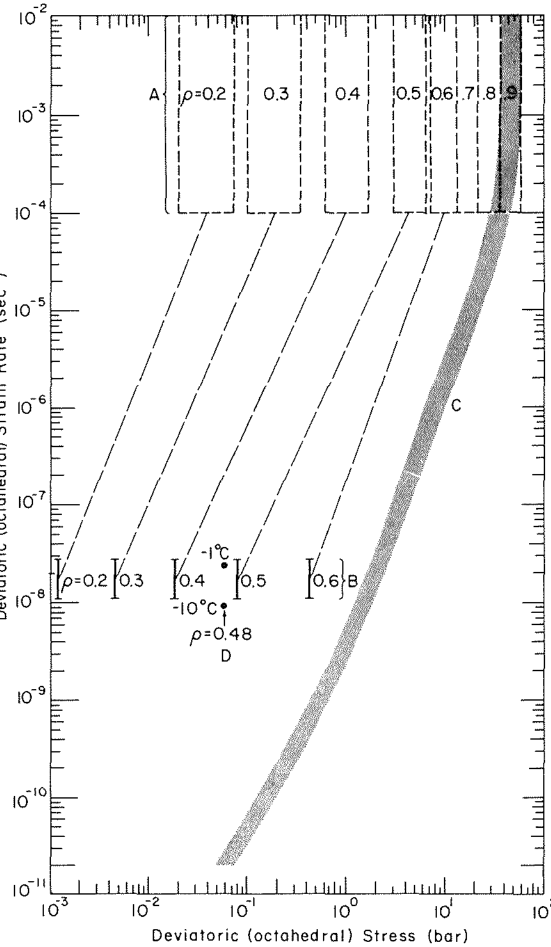


FIGURE 16. Deviatoric stress/strain-rate relationships deduced indirectly from various data sources. (A) Snow, ≈ −10°C (data from Butkovich, 1956; Kovacs *et al.*, 1969; Smith, 1965; Mellor and Smith, 1965; Bucher and Roch, 1948). (B) Snow, ≈0° to −7°C (calculated and extrapolated from data by Haefeli, 1939; Martinelli, 1960; Frutiger and Martinelli, 1966; Judson, pers. comm.). (C) Ice, ≈0.9 g/cm^3, −2° to −10°C (data from Mellor and Smith, 1967; Mellor and Testa, 1969; Hawkes and Mellor, 1972). (D) Snow, 0.49 g/cm^3, −1° to −10°C (Mellor and Smith, 1967).

Constitutive equations for multiaxial stress states

In seeking a general constitutive equation it seems advisable to first consider separately the relationships for deviatoric and dilatational effects. Presently available data are inadequate for this purpose, but some speculation may be helpful for guiding new experimental investigations.

In states of pure shear, snow of constant density seems to be quite similar to ice, in that elasticity may be taken as linear while strain rate for secondary creep is proportional to some power of stress in the ranges that are of most practical interest. For the complete stress range, probable boundary conditions suggest that a simple power law for steady creep is less suitable than a polynomial such as the hyperbolic sine, but in practice this refinement is not of overriding importance. Parameters of the secondary creep stress/strain-rate relation are temperature and density. The temperature parameter has already been covered in the previous discussion of creep, but the earlier presentation of viscosity data perhaps gives a misleading impression of density effects.

From the very skimpy data of Fig. 16, it appears that strain rate for a given stress varies enormously with density; for typical densities the relation can be approximated by a simple exponential of the form $e^{-a\rho}$, in which $a \approx 30\ \mathrm{cm^3/g}$.

For more general shear situations, in which elasticity and viscoelastic creep transients are involved, the stress/strain relations for pure shear become enormously complicated, as can be seen if nonlinear viscosity is considered in the Burgers model. However, there may be cases where an assumption of linear viscosity in a four-element model is supportable if the problem involves a fixed time duration with small creep strains (the situation which gave rise to spurious indications of newtonian viscosity in low stress creep tests on ice).

The volumetric stress/strain relation is even more complicated than the deviatoric one and, of course, volumetric strain strongly influences deviatoric strain rates. The elastic bulk modulus is very strongly dependent on density and therefore on volumetric strain: for snow of density less than $0.6\ \mathrm{g/cm^3}$, bulk modulus increases with the sixth power of density. Bulk viscosity has a particularly complicated dependence on strain, since it is controlled by the interplay of two opposing stress effects. As the stress on typical snow increases, strain rate tends to increase correspondingly, and because volumetric strain is initially achieved by moving grains relative to each other, just as in a deviatoric process, strain rate *tends* to increase with some positive power of the stress. At the same time, resistance to volumetric straining is increasing with strain (and therefore stress) according to some sort of exponential trend. If viscoelastic creep transients were not involved, this effect might be described qualitatively by something similar to

$$\dot{\epsilon} = a\sigma^n e^{-b\sigma} \tag{31}$$

in which a and b are 'constants' related to initial density and other factors, and n is an exponent (>1) identifiable with the power law exponent for deviatoric stress/strain-rate relations.

The prospects for a rigorous volumetric constitutive equation are very discouraging, and it seems inevitable that drastic simplications will be needed for problem solving. In cases involving rapid compression and large strains the answer might be a rate-sensitive equation of state, or a quasiplastic compression modulus. In other cases where total volumetric creep strain is small, it may be feasible to combine bulk viscosity and shear viscosity through simple proportionality, in effect assuming that the viscous analogue of Poisson's ratio remains constant.

The final step in formulating a complete constitutive equation is to combine the dilatational and deviatoric relations into a single stress/strain/time relationship that takes account of viscoelastic behaviour, including nonlinearity of elasticity and viscosity, and is spatially three-dimensional. This development is nowhere in sight, and greatly simplified approximations will have to be devised in accordance with the conditions of each individual problem. However, continued efforts along the lines pioneered by Salm (1967, 1971) seem important as a means of ensuring that simplified constitutive equations are compatible with more general forms.

FAILURE

Failure of snow

The definition of failure in engineering materials is largely arbitrary, since it merely refers to the conditions under which performance ceases to be satisfactory. It may involve rupture, loss of load-bearing capacity, excessive strain, or excessive strain rate. In snow, failure could be actual separation under shear or tension, acceleration of deviatoric strain rate by onset of tertiary creep or by stress increase, volumetric collapse or acceleration of volumetric strain rate, or other effects. This uncertainty carries over into the definition of strength.

For present purposes strength will be regarded as time-dependent, and related to the maximum deviatoric stress that can be reached at a given strain rate, irrespective of whether rupture actually occurs within a limited period of observation.* This does not completely correspond to Salm's (1971) definition, but it permits unified discussion of both brittle and ductile strength failures.

In multiaxial stress fields it is necessary to consider how the interaction of stress components affects failure and strength, and this necessitates a failure criterion, which may be broadly described as a critical functional relationship between the principal stresses for the conditions producing failure. Failure criteria, which may be either empirical or derived from physical models, have variously postulated constancy of maximum principal stress, maximum shear stress, maximum strain energy, and maximum shear strain energy. For snow, Salm (1971) proposed addition of critical power as a criterion, taking into account total elastic strain energy and dissipation rate for viscous work. Brown *et al.* (1973) were more ambitious, attempting to define the conditions for failure by integration over the entire strain history for isothermal deformation according to a fairly general constitutive equation. The Drucker–Prager criterion, a generalized form of certain classical criteria (von Mises, Tresca, Mohr–Coulomb), has attracted interest in snow mechanics (Mellor, 1968; Salm, 1971), since it separates out the effect of the isotropic bulk stress. The Drucker–Prager criterion reduces to the Mohr–Coulomb criterion for the case of plane deformation, and the latter has been used in snow mechanics since the earliest days of the subject.

Failure under uniaxial stress

The most unambiguous determinations of strength are obtained from direct uniaxial stress tests in tension or compression, which do not call for any knowledge of rheological properties in their interpretation. However, while these tests are simple in principle they can be complicated in practice, especially where high strain rates and brittle fracture are concerned (Hawkes and Mellor, 1970). There are other tests which purport to measure uniaxial strength indirectly by invoking rheological properties for the material, usually an assumption of linear elasticity. They include the ring tensile test, which has been shown to be invalid for most engineering materials (Mellor and Hawkes, 1971), and beam bending tests, which also seem dubious for highly compressible viscoelastic materials.

A general impression of the uniaxial strength of bonded dry snow is given by Fig. 17, which shows tensile and compressive strengths as functions of density for relatively high strain rates (10^{-4} to $10^{-2} s^{-1}$). Broadly speaking, tensile and compressive strength are equal at low densities, while at the density of solid ice the ratio of compressive strength to tensile strength is about 5 on this plot. Brittle fracture of porous solids (rocks, ceramics, concrete, etc.) is often analysed in terms of Griffith fracture theory and its later derivatives, using failure criteria that imply compression/tension strength ratios ranging from the classical Griffith value of 8 to values between 10 and 20. The data of Fig. 17 might be regarded as characterizing brittle fracture, i.e. creep strain is small relative to elastic strain for the loading that leads to failure. As strain rates drop (or temperatures increase) so that failure becomes ductile, i.e. creep strain becomes large relative to elastic strain, then strength (as defined earlier) can be expected to decrease significantly for very dense snow. This relationship has been given for a very wide range of strain rates in ice (Hawkes and Mellor, 1972), but a different situation prevails for slow compression of typical snow, which increases in density and effectively work hardens.

* This implicitly identifies the secondary creep inflection point of a complete classical creep curve with the peak of a conventional stress/strain curve. In either case the time taken to reach this point is what is sometimes called 'time-to-failure', which could be effectively infinite in some practical problems.

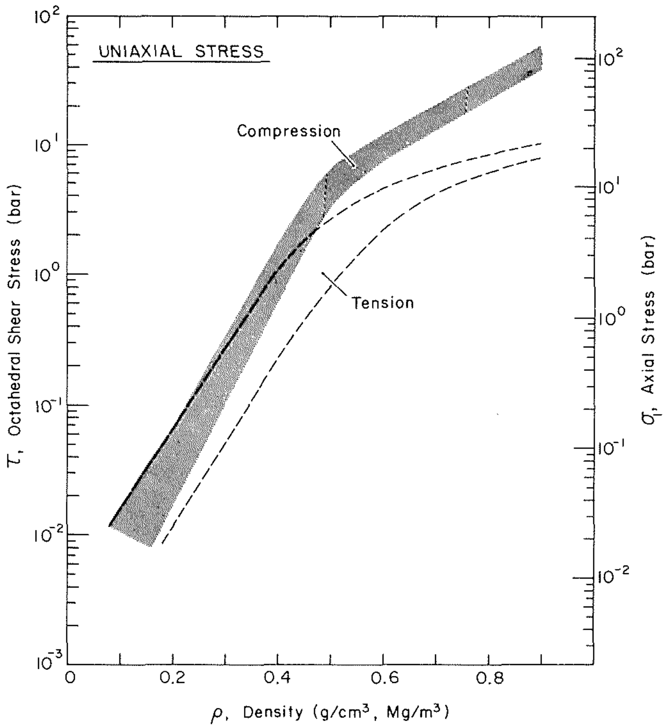


FIGURE 17. Strength of dry, coherent snow under rapid loading in uniaxial stress states. (Data from Bucher, 1948; Butkovich, 1956; Haefeli, 1939; Hawkes and Mellor, 1972; Keeler, 1969; Keeler and Weeks, 1967; Kovacs *et al.*, 1969; Mellor and Smith, 1965; Ramseier, 1963; Smith, 1963; Smith, 1965.)

One interesting and somewhat controversial point about the rate sensitivity of strength concerns the apparent small drop in compressive strength as strain rate increases through the ductile/brittle transition range. In the case of ice it has been shown that, at least in some cases, the drop was caused by poor testing technique for the brittle range (Hawkes and Mellor, 1972). However, during the course of a test there is viscous energy dissipation which must produce temperature rise in all but very slow tests, and in principle viscous heating should lower the apparent strength. This effect, which is an intrinsic part of Salm's (1971) theory, does not appear capable of producing significant homogeneous temperature increases in typical ice test specimens, but if temperature rise is concentrated at active microdeformation sites (grain boundaries, defect structures) then the effect may well be significant, especially in snow.

Many experimenters, including the writer, have attempted to define the effect of temperature on strength, but presently available data are not very convincing. It seems likely that data for high temperatures (above −10°C) actually reflect temperature/strain rate interactions that are not adequately defined. In the brittle fracture range, strong temperature dependence is not to be expected, whereas in the creep rupture range the temperature relationship should be similar to that which controls sustained creep.

Shear strength

The term 'shear strength', which is subject to varied interpretations, tends to generate confusion at the best of times, but in the case of highly compressible snow it can be very confusing indeed. One measure of shear strength can be obtained from uniaxial stress tests, in which maximum shear stress is $\sigma_1/2$ and octahedral shear stress is $\sigma_1/\sqrt{3}$, σ_1 being the axial stress. However, the general practice has been to define and determine shear strength according to the Mohr–Coulomb failure criterion for biaxial stress fields:

$$(\sigma_1 - \sigma_2) - (\sigma_1 + \sigma_2) \sin \phi = 2c \cos \phi \tag{32}$$

or

$$s = c + p \tan \phi \tag{33}$$

where σ_1 and σ_2 are principal stresses, c is cohesion, ϕ is the angle of friction, s is shear strength, and p is normal stress.

The idea of characterizing snow as a $c - \phi$ material is attractive for some purposes, as c can be identified with the time-dependent (sintered) intergranular bonding, and ϕ can be thought of as characterizing initial strength for disaggregated snow and residual strength after bonds have been broken. There may be cases involving reworked snow (e.g. 'Peter snow') where this concept is useful, but in general it could be deceptive. It is questionable whether internal friction can be fully mobilized until c is effectively destroyed, either by deviatoric stress or bulk stress (p), and there is no doubt that in some cases p, or bulk stress, completely changes the state of the material, thereby vitiating the concept.

Some of these ideas can be explored with the aid of data from careful direct shear tests by Ballard *et al.* (1965), who tabulated their results but did not undertake any analysis. The data indicate that c is virtually zero at the beginning of the sintering process for moderately dense (0.36–0.59 g/cm^3) snow. For small values of p, s increases by over an order of magnitude after a long period of sintering, while $p \tan \phi$ increases by factors between 2 and 5, depending on temperature. For large values of p, s is not much greater than $p \tan \phi$ at any stage of sintering, indicating that c is largely destroyed by the test. This is confirmed by records of density before and during the tests, which show that some specimens suffered as much as 64 per cent volumetric strain.

Some of the shear strength presentations in the literature give a misleading impression, and it is therefore pointless to summarize them here. However, Fig. 18 gives a general idea of the magnitude of shear stress for small values of p as measured by shear boxes, shear cylinders, shear vanes, torsion tests, and avalanche fracture line analyses. The values are in broad agreement with those that would be deduced from the uniaxial strength data of Fig. 17.* Certain results obtained with *in situ* rotary shear vanes were rejected as anomalous in preparing this plot, but it was gratifying to find that analyses of avalanche fracture line profiles appeared to give consistent data.

Failure theories and failure criteria

In materials science, both physical and statistical failure theories are based largely on the idea that strength is governed by the nature, form, size and distribution of flaws or structural defects. For example, creep theory is heavily influenced by ideas of dislocation motion, brittle fracture theory is concerned with stress concentrations at internal microcracks or other flaws, and statistical strength of materials (Weibull theory) treats the probability aspects of flaw size distributions. These ideas have been applied convincingly to ice of low porosity, but not to snow.

Ballard and McGaw (1965) took a different approach, ignoring stress concentration and regarding pore space as effective loss of load-bearing area, thereby obtaining a

* The shear values for low density snow are surprisingly small when compared with values for uniaxial tensile strength. Some direct correlation studies on low density snow have indicated that centrifugal tensile strength is 10 times higher than shear strength measured by vanes or boxes.

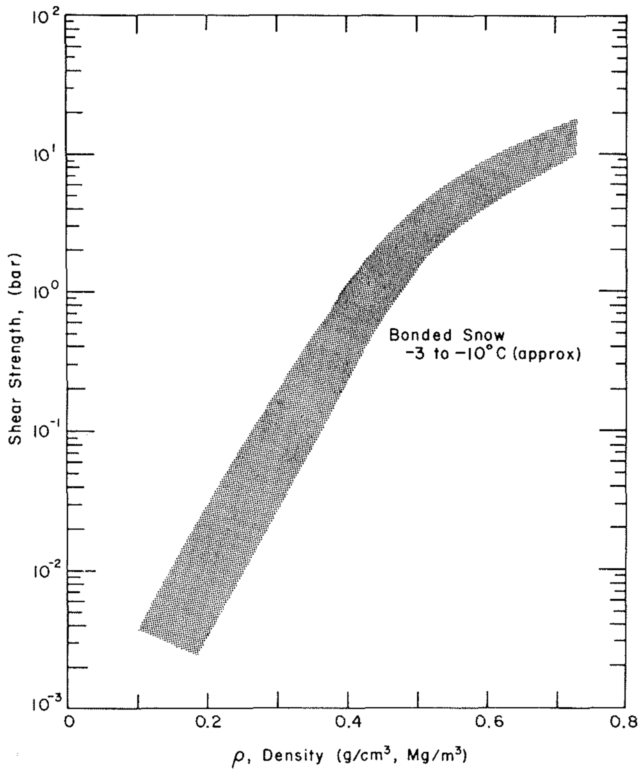


FIGURE 18. Shear strength of dry, coherent snow from direct measurements. (Data from Ballard *et al.*, 1965; Butkovich, 1956; Haefeli, 1939; Keeler, 1969; Keeler and Weeks, 1967. Also avalanche release data from various sources summarized by Keeler (1969). Some values from *in situ* shear vanes were rejected.)

linear relation between strength and effective porosity on potential failure surfaces:

$$\sigma_n/\sigma_0 = 1 - n/n_* \tag{34}$$

where σ_n is failure strength at porosity n, σ_0 is strength at zero porosity (solid ice), and n_* is a porosity intercept representing the porosity at which strength drops to zero. They deduced values of n_* from 0.49 to 0.57 (corresponding to densities of 0.47–0.39 g/cm^3), and took values of σ_0 that were far too small. A more realistic nonlinear empirical relationship between relative strength and porosity has been adopted from the technology of ceramics and sintered metallic compounds by Gow (1974). This relation is in fair agreement with the experimental data for snow denser than about 0.5 g/cm^3, but it is necessarily restricted to densities above about 0.3 g/cm^3, since that corresponds to the highest porosity that can be achieved with equant grains.

The variation of relative strength according to porosity is relevant to well-bonded snow, but not necessarily to cohesionless snow. For the latter case, empirical relations between friction angle ϕ and porosity n can be developed from data for soils and other granular materials; for equant grains, ϕ drops to small values at porosities above 0.5, i.e. at snow densities below about 0.46 g/cm^3.

In some areas physical failure theories (e.g. Griffith theory) lead to failure criteria, but at the present time there do not appear to be failure theories for snow that can perform this function. It is therefore necessary to rely solely on phenomenological failure criteria.

The classical phenomenological criteria are clearly unsuitable for general application to snow, although there are some that may find restricted application in special problems, e.g. in cases of rapid loading where deviatoric stress exceeds bulk stress. The failure criterion for snow proposed by Brown *et al.* (1973) is very complex, but still it skirts around the essential characteristic of high compressibility. The one-dimensional criterion developed by Salm (1971) shows a great deal of promise, but it is still far from ideal generality. The unavoidable conclusion is that, in the immediate future, simplified failure criteria will have to be improvized to suit particular cases.

Collapse mechanics

When a plate is pressed normally into the surface of a semi-infinite snow mass, the motion may involve a series of discontinuous collapses. Study of this effect has been referred to as 'collapse mechanics', and it has attracted a certain amount of interest, especially with reference to critical penetration speeds. While this is a particular type of boundary-value problem, and therefore outside the scope of this general review, it seems worthwhile to point out that the nature of the observed phenomena is partly controlled by the experimental equipment. With a perfectly stiff loading system giving penetration at constant speed, the force can drop almost instantaneously to zero after a sudden collapse, while with a perfectly compliant system the force can never drop, even though there is instantaneous increase in displacement after each collapse. Unless the importance of loading system compliance is understood, this topic can be confusing. Certain types of strength-measuring penetrometers can produce intermittent collapse, and with indeterminate compliance and uncontrolled inertial effects test results may give a false impression.

Arbitrary strength measurements and ram resistance

Over the years there have been a fair number of devices that purport to measure snow strength but give results in arbitrary terms, some of them rather mystical. Of these, the most important is the Rammsonde, which is now challenged by its automated offspring, the resistograph. Most of the data collected by arbitrary devices are unusable in formal mechanics, but a special effort to interpret Rammsonde data seems justified in view of the widespread and longstanding use of the device.

Rammsonde penetration resistance R has the dimensions of force (kgf), but it actually represents driving energy per unit penetration (cm-kgf/cm) for a fixed cross-sectional area of penetrometer point A (usually 12.6 cm^2), and therefore R/A can be interpreted as a specific energy for penetration (cm-kgf/cm^3). In snow of typical densities (say up to 0.6 g/cm^3), the energy is expended in breaking bonds, displacing material laterally, and compacting in an annular zone around the penetrometer hole. If KA is effective cross-sectional area for the total disrupted zone, specific energy for the disruption process would be R/KA, and K would be expected to increase with density from a lower limit of unity. In the field of rock mechanics it has been shown that the uniaxial compression test provides a standard measure of specific energy for brittle fracture that can be used as a basis for normalization of process specific energy values (Mellor, 1972), and thus there are sound reasons for seeking correlation between ram resistance and uniaxial compressive strength in snow.

Keeler and Weeks (1967) and Keeler (1969) sought simple linear correlations between strength S and ram resistance R, while Abele (1963) based his correlation on a simple exponential, $R = ae^{bS}$, which seems more reasonable with respect to probable boundary conditions for the complete range of density. However, on the basis of available data neither correlation seems to have general validity. Shear strength data were used by Keeler and Weeks, while Abele worked with uniaxial compressive strength, but if these quantities are assumed to be related by a factor of 2 it is found that neither log–log nor semilog plots come anywhere near linearizing the complete range of data, so that linear regression analysis cannot be usefully performed on the logarithmic forms of either simple power relations or simple exponential relations. From the argument of the previous paragraph, a linear correlation would be expected between S and R/KA, but it appears from the data that K must be a complicated function of snow density. For the time being, the only sensible course is to compile completely empirical correlations in direct graphical form.

Indirect estimation of strength

When materials fail by brittle fracture there is usually a linear correlation between uniaxial compressive strength σ_f and Young's modulus E, i.e. failure strain is roughly constant. For rocks that are not strongly anisotropic, σ_f/E lies between 2×10^{-3} and 5×10^{-3}, for frozen sand σ_f/E is from 2×10^{-3} to 3×10^{-3}, and for polycrystalline ice σ_f/E is about 2×10^{-3} (Mellor, 1972). When the stress/strain plot is nonlinear, E is usually taken as the tangent modulus at 50 per cent of ultimate stress.

If a similar correlation can be demonstrated for snow, then strength can be estimated from measured values of Young's modulus. This would be useful if E could be measured *in situ* by nondestructive acoustic methods, especially in situations where remote sensing of strength is attractive.

From comparison of Figs. 2 and 17 it can be seen that σ_f/E is approximately constant for the complete range of snow density, with values of 2×10^{-3} or greater when static values of E are used, and values around 1×10^{-3} when dynamic values of E are used. In compression tests on dry snow by Kovacs *et al.* (1969) the failure strain, which represents the ratio of failure stress to *secant* modulus, ranged from 1×10^{-3} to 6×10^{-3}. Thus it seems likely that there is a linear correlation between σ_f and E for dry snow, so that strength can be estimated on the basis of modulus if the regression line is established by direct experiment.

BOUNDARY FRICTION

Surface friction and adhesion

Although surface friction does not belong in the same category as bulk mechanical properties, it is important in defining boundary shear stresses in a variety of problems that involve both coherent snow and fluidized snow.

Abnormally low surface friction is one of the more useful and agreeable properties of snow, but its nature is not well understood. Part of the difficulty is that elementary friction theory is one of those subjects that traditionally have been taught in terms of oversimplified physical principles (the competition driver soon repeals the laws of Amontons and Coulomb, just as the aerobatic pilot manages to confound Bernoulli). Another source of confusion is the lumping of form drag and sinkage resistance with friction (in the following notes it is implicit that any kind of macroscopic snow displacement, irreversible compression, or other inelastic bulk processes that dissipate energy are excluded from the friction process). However, the main problem is that the physics of friction on snow and ice has not been adequately dealt with.

Kinetic friction of coherent snow

Interfacial friction between solids is sometimes referred to as 'dry friction', but in the case of kinetic friction on snow there is usually water at the sliding interface, since frictional heating can produce the small temperature rise needed to cause melting. The presence of water, which has been adequately demonstrated by electrical conductivity measurements and experiments with water-soluble dyes, was formerly thought to be the result of pressure melting, but it is now generally considered that simple pressure melting can only be significant at temperatures very close to 0°C. Whatever the ideas on melt mechanisms, water lubrication has long been thought to explain most of the peculiarities of sliding friction for snow and ice, and this theory still finds wide acceptance (Bowden, 1955; Bowden and Tabor, 1950, 1960, 1964; Shimbo, 1971), although quantitative analytical treatment has not been applied to any great extent. However, the Bowden theory has been questioned (for example by McConica, 1950, and by Huzioka in a long series of papers published in *Low Temperature Science*) and a more carefully argued alternative has been put forward by Niven (1954, 1956a, b, c, 1959, 1963), who appears to have convinced J. D. Bernal by means of a critical experiment on bismuth. Niven's theory deals with the physical chemistry of surfaces, and the low friction of ice is explained by the facility with which the open tetrahedral molecular structure can reorient to permit plastic flow of surface asperities when thermally activated.

In many ways the Bowden and Niven theories are similar for practical purposes, since complete molecular reorientation as envisaged by Niven produces the structure of liquid water, and we are simply left with a semantic confusion over the meaning of 'lubrication'. The main difference ought to lie in the response to temperatures below the melting point, when no water is formed but plastic flow of microscopic asperities is temperature dependent.

According to the classical definition, the coefficient of kinetic friction μ_k is the ratio of tangential to normal *force,* irrespective of the size of the bearing area, the interfacial contact pressure, the sliding speed, or the ambient temperature. In the case of snow and ice, it is to be expected that the classical laws of friction will be violated, at least in principle, since interfacial temperature rise depends on the rate and duration of both heat generation and heat loss. For a given sliding velocity, initial heat generation rate per unit surface area should be proportional to the frictional stress (and therefore to the mean contact pressure), and the duration of heating should be proportional to the length of the contact area in the direction of travel. With constant temperature boundary conditions and given thermal properties for the materials, initial temperature rise should thus be proportional to contact pressure and slider length. This implies that for geometrically similar sliders carrying a given fixed weight, temperature rise should be inversely proportional to slider length. When sliding velocity varies, this simple model would predict an initial temperature rise proportional to velocity if no additional cooling effects are introduced. However, the experimental evidence bearing on these matters tends to be confusing, as the interrelated variables are not always adequately controlled or separated. Neither Bowden and Tabor (1964) nor Shimbo (1961, 1971) detected significant variation of μ_k with pressure in a limited range (< 0.1 bar), but the 'melting' was probably adequately established in their tests. On the other hand, Niven (1959, 1963) found that frictional resistance on cold ice became almost constant as contact pressure increased from about 15 to 30 bars. Ericksson (1949) found that μ_k for steel dropped by about one-third as pressure increased from about 0.01 to 0.7 bar on 0°C snow, and by one-third to one-half as pressure increased from 0.2 to 2 bars on ice at temperatures from 0° to −18°C (another experiment which varied area with fixed weight gave consistent results). Klein (1947) showed μ_k for brass, Bakelite, and waxed wood dropping by 10–20 per cent as pressure increased from 0.05 to 0.24 bar on cold snow. For plastic sliding on ice at −15°C, McConica (1950) found μ_k increasing with pressure below 0.05 bar but decreasing towards an asymptotic limit above 0.08 bar. Ericksson also slid geometrically similar (20:1) steel runners on −4°C ice with a pressure of 0.3 bar and found that μ_k dropped by two-thirds as runner length increased from 0.1 to 1.7 m. The general impression is that μ_k tends to decrease with increasing pressure and with increasing runner length even after 'melting' has been initiated, perhaps because these factors work towards an increase in the number of sites at which favourable conditions can develop.

Pressure effects should have some bearing on the question of possible variation of

μ_k with snow density and grain size. For a given nominal contact pressure, actual mean pressure must increase with surface porosity, while the size of grains and pores should have some effect on local heat dissipation and on the molecular structure of the ice surface. There is not much experimental evidence on which to base speculation. Shimbo did not find any significant difference between compacted snow and crushed ice at -1° to 0°C when sliding dense polyethylene and ski wax surfaces at low contact pressures. On the other hand, Ericksson found that μ_k for steel (0.3 bar, -2° and -25°C) increased by a factor of 3 as effective grain size decreased from infinity (solid ice) to about 0.15 mm. The latter experiment mixes the influence of grain size and porosity, but it tends to suggest a dominant grain size effect, since porosity cannot vary greatly with random packing of equant grains.

Still following up ideas of effective bearing area, microroughness of the slider surface should have some influence, but again the experimental evidence is somewhat confusing. Shimbo tested slightly roughened phenolic resin sliders on wet snow, the rationale being that roughening would break the 'viscous drag' of a continuous water film. As roughness increased from 'zero' to 0.015 mm, μ_k dropped by a factor of 2, but further increase of roughness to 0.035 mm had no significant additional effect. Ericksson obtained conflicting results with steel surfaces on snow: at 0°C a polished and chromed surface gave the lowest sliding friction and a roughly filed surface gave the highest friction, with an ordinary machined finish in between. However, below about -3°C the order was reversed, and at -29°C the polished surface had double the sliding friction of the rough filed surface. Nominal contact pressure was 0.3 bar, which is about an order of magnitude higher than the pressure used by Shimbo.

Speed of sliding ought to affect μ_k, as it controls the rate of frictional heat generation, which must exceed the rate of heat transfer from generation sites if interfacial temperature is to reach 0°C. Power density at the sliding surface is given by the product of frictional stress and sliding velocity. If frictional stress were invariant with velocity, as might be approximately true in the initial stage of sliding, the rate of heat generation per unit surface area would be proportional to velocity. However, if frictional stress were itself inversely related to velocity, as might happen once 'melting' was initiated, there would be a more complicated feedback effect.
In experiments on snow at -10°C, Bowden (1955, 1964) found that μ_k for aluminium, Perspex, lacquered wood and waxed wood stayed at relatively high values ($\approx$0.2 to 0.4) until the sliding speed exceeded 0.03 m/s; thereafter, μ_k dropped until all values were an order of magnitude smaller at 5 m/s. McConica (1950) slid Bakelite on ice at -15°C and found that μ_k decreased up to a speed of 8 m/s. Kuroiwa *et al.* (1969) tested plastic-base skis on packed snow; at 0°C, μ_k increased gradually, doubling or trebling as speed increased from 7 to 21 m/s, while at -1.6 to -2.5°C, μ_k stayed fairly constant from 7 to 17 m/s and then doubled as speed increased further to 21 m/s. Shimbo studied speed effects on wet snow, which is not dependent on frictional heating for 'lubrication'. There is a factor of 10 discrepancy between text and graphs in his papers, but it seems that the small speed effect for PTFE became completely insignificant above about 0.05 m/s. Bowden also found that on slowing down after a high-speed run, cold snow adheres to surfaces that are not strongly hydrophobic, apparently because the meltwater film refreezes.

Temperature has a marked effect on sliding friction. For metals and hard plastics sliding at low speed, μ_k increases approximately linearly with decreasing temperature within the common range of temperatures for dry snow. Ericksson found μ_k for smooth steel approximately doubling as temperature dropped from 0 to -25°C, Klein showed μ_k for brass, Bakelite and beeswax increasing by factors of 3–4 over the same temperature range, and Bowden showed an increase by a factor of 3.5 for PTFE. At very low temperatures (say below -40°C) with common sliding speeds (say up to about 15 m/s) it appears that interfacial 'melting' ceases to occur (e.g.

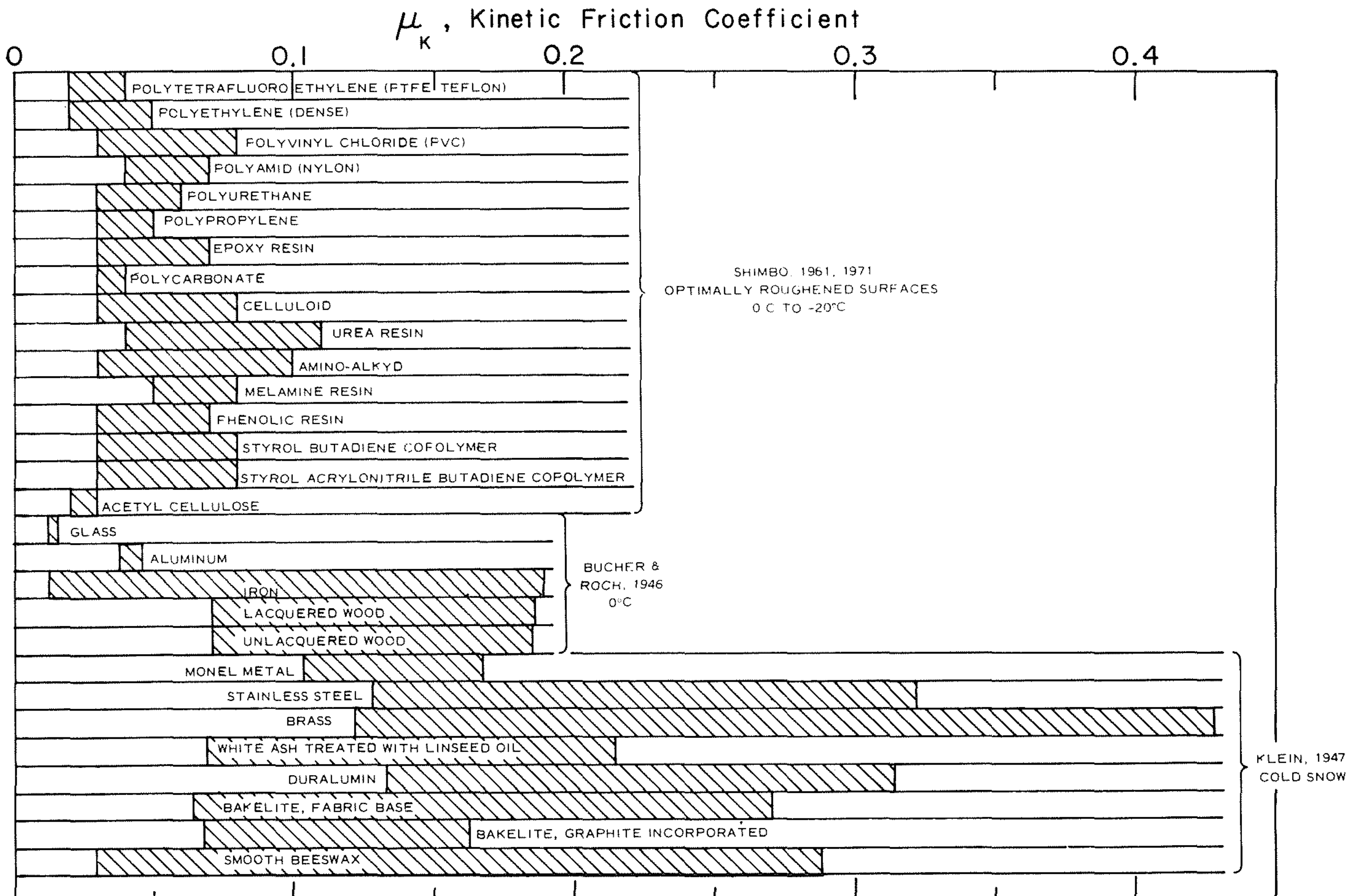


FIGURE 19. Representative values for coefficient of kinetic friction of various substances sliding on snow at moderate speeds and typical ambient temperatures (high limit values for some materials result from wetting effects).

Budnevich and Deriagin, 1952; Goldovskii, 1953), and μ_k for snow becomes very similar to values for dry sand. However, with large skis sliding at very high speeds it may still be possible to raise interfacial temperature to the melting point even though ambient temperatures are very low (the South Pole airstrip becomes glazed after frequent use by C-130 cargo aircraft, which have large skis that slide under almost full load at speeds around 50 m/s). Surfaces coated with ski waxes and lacquers may respond to temperature in a different way, with extreme and abrupt changes in μ_k within certain temperature ranges, presumably because the properties of the coatings themselves change with temperature.

In very wet snow there is an abundance of interfacial water, and effective frictional resistance may increase if the slider is easily wetted. Some hydrophilic materials (e.g. aluminium, nylon, Perspex and some lacquers) have zero contact angle with water and are therefore completely wetted, while at the other end of the scale there are strongly hydrophobic materials that resist wetting and tend to have low friction under all water-lubricated conditions; the outstanding example is PTFE, with a contact angle of 126° (Bowden and Tabor, 1964).

Although snow surfaces provide relatively low sliding friction for a wide range of materials, there are significant differences in μ_k for various materials. In general, it appears that substances which develop only weak molecular forces at the interface, as indicated by high contact angle, tend to be superior under all conditions. The penetration and flow resistance (hardness) of the slider surface is probably also important, but there does not appear to be a simple relation between hardness and friction. Recent studies indicate that low friction polymers shed thin layers to the substrate on sliding, and at the same time acquire orientation of the surface structure.

Some representatives values of μ_k for various materials are given in Fig. 19.

Static friction and adhesion

Static friction on snow and ice should probably be defined as the 'isothermal' limit of kinetic friction as sliding velocity tends to zero. If a slider stands still for a finite length of time on 'cold' snow or ice there will usually be progressive development of interfacial bonds that give *tensile* connection across the interface, and this connection must be regarded as adhesion. Adhesion is proportional to interface area, and therefore is quite different from friction as described by the Amonton–Coulomb laws. This very fundamental distinction has not been adequately recognized, and experimental data tend to be confusing.

The zero speed limit of sliding friction is not easy to measure directly with simple equipment, but for dry snow it should be approximately equal to the value measured for low sliding speeds and very low temperatures, i.e. the 'unlubricated' value, which seems to be quite similar to the sliding friction coefficient for dry sand. For snow at 0°C there should not be much difference between the coefficients for sliding friction and static friction if surface tension effects in the interfacial water are small.

Another possibility for measuring static friction (which does not seem to have been tried) would be to measure interfacial shear resistance as a function of contact time and obtain true static friction from the zero time intercept. With this method it has to be recognized that true static friction is not strongly dependent on contact pressure and bearing area, while adhesion is proportional to area.

A discussion of ice adhesion is beyond the scope of this review, but the general features of adhesion seem fairly straightforward. The strength of the adhesive bond between cold snow and a solid surface can be expected to increase exponentially with time from a lower limit representing true static friction to an asymptotic upper limit reached after interface equilibration by molecular orientation and diffusion. The rate constant for time-dependent bonding can be expected to be influenced by temperature (initial and final), temperature gradients, contact pressure, size and shape of snow grains, and surface properties of the solid material. The limiting bond strength will usually increase with decreasing temperature, and it will certainly vary with the surface properties of the solid material, strongly hydrophobic materials with high contact angles providing the weakest bonds. An upper limit to shear resistance is set by the shear strength of the snow beneath the slider: if the bond strength is less than the strength of the snow, shear displacement will occur at the interface, but if the bond strength is greater than the strength of the snow, shearing will occur in the snow beneath the interface.

Presently available data for static friction and adhesion on snow do not inspire much confidence, but Fig. 20, which shows low temperature values for low-speed sliding friction, probably gives a fair impression of true static friction.

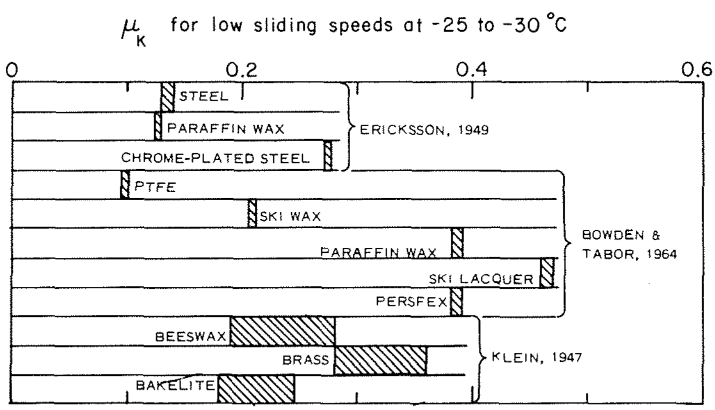


FIGURE 20. Representative values for coefficient of kinetic friction of various substances sliding on snow at low speeds and low ambient temperatures (–25° to –30°C).

Friction, or boundary shear stress, for fluidized snow

When the intergranular bond structure of snow is destroyed and effective contact pressure between grains becomes small, the material is fluidized, i.e. it loses its ability to resist shear. Fluidization occurs when snow is blown by the wind (or by pumps), when loose or pulverized snow avalanches, and when snow is violently disturbed by slipping vehicle tracks or snowploughs. In the fluidized state, shearing takes place within the snow mass and not at the snow/solid interface, but a shear stress, or 'wall friction', is transmitted to the solid. The magnitude of this boundary shear stress is determined by the effective viscosity of the fluidized snow (including the eddy viscosity in turbulent flow) and by the velocity gradient in the boundary layer.

The properties of fluidized snow do not seem to have been studied, and no data are available. However, the problem can be illustrated and explored by consideration of very low density material such as exists near the solid boundary during major blizzards and during dust avalanches.

In very low density dispersed snow the newtonian viscosity must be close to that of air. H. A. Einstein's expression for the viscosity of low concentration suspensions μ gives the estimate

$$\mu = \mu_0(1 + \tfrac{5}{8}c_v) \tag{35}$$

where μ_0 is viscosity without solids and c_v is volumetric concentration of solids. For well dispersed snow, c_v is probably in the range 10^{-3} (boundary of blizzard or dust avalanche) to 10^{-1} (equivalent to density of 0.1 g/cm^3), and therefore the second term of equation (35) cannot have any significant effect (for air, $\mu_0 \approx 1.7 \times 10^{-4}$ poise). With turbulent flow in the boundary layer inertial effects dominate and a snow suspension may behave in approximately the same way as a clear newtonian fluid, in which case the boundary shear stress τ can be expressed as

$$\tau = (\mu + \mu_e)\,\frac{du}{dz} \approx \mu_e\,\frac{du}{dz} = \rho\epsilon\,\frac{du}{dz} = \rho\left(kz\,\frac{du}{dz}\right)^2 = \rho u_*^2 = \rho C_d \bar{u}^2 \tag{36}$$

where u is tangential flow velocity, z is normal distance from the solid boundary, μ_e is the 'eddy viscosity' or momentum transfer coefficient for turbulent mixing, ρ is effective fluid density, ϵ is the kinematic eddy viscosity, k is von Karman's constant (≈ 0.4 for simple newtonian fluids), u_* is the shear velocity or friction velocity, and C_d is a surface drag coefficient. For blizzard winds blowing over flat snow surfaces $C_d \approx 1.3 \times 10^{-3}$ (i.e. $u_* \approx 3.6 \times 10^{-2}\bar{u}$), so that $\tau \approx 3 \times 10^{-12}\,\bar{u}^2$ bar, where $\bar{u}$ is in cm/s and $\rho = 2.3 \times 10^{-3}$ g/cm^3. With speeds of 10 and 100 m/s, shear stresses are 3×10^{-6} and 3×10^{-4} bar respectively. If it is assumed that ρ can be increased without changing the structure of the turbulent boundary layer, then τ would simply increase in proportion to ρ. The static shear strength of low density deposited snow (0.1 g/cm^3) is probably about 10^{-3} bar; this stress level might be reached inertially in a turbulent boundary layer with *snow* density of 10^{-2} g/cm^3 and a speed of 80 m/s.

The above argument is obviously oversimplified and not directly applicable to the shearing of granular masses where grains are in frequent contact with each other. Clearly there is much work to be done in this area.

CONCLUSION

While it may be axiomatic that all materials possess all rheological properties, there is no material of broad engineering significance that under normal conditions displays the bewildering complexities found in snow. If constitutive equations and failure criteria could be formulated with complete generality for snow, they would probably cover all contingencies for all real solids. Recognition of this puts the basic problems of snow mechanics into perspective; it seems unrealistic and presumptuous to immediately seek complete generality when much simpler materials are presenting formidable problems in other branches of solid mechanics.

For the time being it will be necessary to adopt greatly simplified rheological descriptions for snow, concentrating on the characteristics that dominate for particular problems. At the same time, the general behaviour of snow has to be kept in mind so as to avoid unrealistic distortions in constitutive equations and failure criteria. Elegant simplification of complicated behaviour is very much needed.

Laboratory tests on low density snow tend to be difficult and expensive, and the possibilities for acquiring data from field observations and *in situ* measurements need to be fully exploited. Useful compressibility data are obtainable from depth/density characteristics, deviatoric stress/strain-rate relations can be deduced from downslope creep profiles, shear strength can be estimated from analysis of avalanche starting zones, and strength can also be estimated on the basis of *in situ* acoustic measurements. Statistical validity is inherent in these methods, but refinements such as multivariate analysis are probably needed to account for the effects of uncontrolled variables. Additional field data sources (e.g. 'layer profiles' giving volumetric strain as a function of time) could be developed.

Boundary friction, a topic outside the mainstream of snow mechanics, offers some potential for further research. Definitive experiments on kinetic friction of coherent snow would be useful, and more work on static friction and adhesion would have practical value. Boundary shear stresses developed by fluidized snow are almost totally unknown.

Past research in snow mechanics has involved diverse methodologies from a variety of academic disciplines, with a wide range in the level of sophistication. While stimulating at the research level, this could make the subject somewhat confusing for non-specialist engineers and mechanicians, and close conformance to the methodology of classical mechanics seems desirable.

Snow mechanics began as a practical science that was intended largely to improve engineering capabilities, and most current research is supported on the assumption that it will produce useful results. Thus, while recognizing the necessity for more abstract academic research, the main efforts in the field should probably be in pursuit of useful results that are comprehensible to nonspecialist engineers and technologists.

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DISCUSSION

W. Good:
Perhaps some of the scatter could be removed if mechanical properties are considered functions of snow texture as well as density.

R. A. Sommerfeld:
Much of Keeler and Weeks' (1967) tensile strength measurements fall above your upper envelope.

W. St Lawrence:
We have not been able to verify that the viscous analogue of Poisson's ratio in tension is 0.5.

Malcolm Mellor:
I made an attempt to summarize, as simply as possible, broad magnitudes and trends of available data. If texture could be introduced systematically, scatter and inconsistencies would indeed be reduced.

J. F. Nye:
Dr Mellor's extremely useful and balanced paper gives us the opportunity to discuss the major question of where the whole subject of snow mechanics is going. It seems to me that there may be a lack of proportion in some of the papers presented at this symposium. We see attempts to fit experimental results with rather large scatter to precise and sophisticated constitutive laws. I would like to underscore the statement made in Dr Mellor's paper: 'Elegant simplification of complicated behaviour is very much needed.'