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New methods of measurement of slow particulate transport processes on hillside slopes

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ABSTRACT Two new methods are proposed. The one relies on a steady state solution to the diffusion equation for the region bounded internally by a circular cylinder and the other upon a method of estimating the velocity of particles by counting.

Méthodes nouvelles de mesure du transport lent des particules sur les pentes des collines
RESUME Deux méthodes nouvelles sont proposé. L'une repose sur une solution a régime permanent de l'équation de la diffusion pour la région bornée par un cylindre circulaire et l'autre sur une méthode pour évaluer la vitesse des particules par décompte.

INTRODUCTION

Present methods of measuring soil transportation rates suffer from three main disadvantages. Soil creep in the geomorphic sense proceeds very slowly and direct measurement necessitates lengthy intervals between observations. Increasing instrumental precision acts to cut down the interval but brings in its train the further problem that measurements then tend to pick up a great deal of noise, so requiring a lengthened series of observations for an adequate sample. Secondly, probes, pits and allied techniques interfere with the flow they purport to measure. Finally, present methods treat the soil as if it were a continuous medium. They fail to realize the detail of an essentially particularate flow; at best they integrate over an area or a depth to provide some form of mean value. The use of radioactive tracers has up to the present been disappointing, not only because of technical difficulties and a naive method of sampling but even more so because of the lack of a predictive theory and a consequent reliance on elementary hydrological analogies.

The problem is to devise a method of unambiguously measuring soil creep at the particle level without interfering with the soil structure or the transport processes, or alternatively, requiring just one observation, which can therefore be destructive. We outline two partial solutions still being developed; a complete solution obviating all the disadvantages listed is probably unattainable until we have resolved the difficulties attendant upon the use of radioactive tracers. The one involves a steady state solution which ideally requires only one observation, the other estimates particle velocities by simple counting and eliminates the need for elaborate procedures and hardware. Both approaches involve long and involved mathematical arguments to be published in full elsewhere.

The theory of particle transport on soil covered slopes set out in Culling (1963), provides a link between the fundamentally reversible behaviour of soil particles and the essentially irreversible development of hillside slopes. The soil is envisaged as an aggregate of particles which in most cases act as individual bodies. Due to the unlimited variation in the orientation of planes of inter-particle contact a pure normal or shear stress applied at a boundary will be resolved at any interior contact into both types of component with no restriction on the partition. Thus a force tending to displace an individual particle can take up any direction, though some directions may be preferred. The effect upon the particle is in almost all cases qualified by the local environment, in particular by the distribution of pore space and the resistance of neighbouring particles to dilatency. Thus any forces whether arising internally or from external sources, whether operating upon individual particles or upon the aggregate as a whole will if they produce a displacement of soil particles include in any translation a random component. Soil transport processes therefore depend upon the activity of the particles and on the restrictions to free movement imposed by the soil structure. In general the resulting macroscopic process will consist of a directed drift component and a diffusive random component. The presence of this latter component for which standard Brownian motion is the classical model renders the estimation of soil particle velocities much more than a simple time and distance observation.

STEADY STATE SOLUTION

This stems from classical sedimentation theory. If a finite amount of recognizable material is introduced to the soil aggregate at a specified point on a uniform hillside slope, the particles are subject to a drift downslope plus dispersion due to diffusive movements representing the random component. Measurement of the drift and the dispersion provides estimates of the diffusion and drift coefficients. This of course takes time. If however, the introduced material meets a reflective barrier the concentration of particles will eventually settle down to a steady distribution. The solution of this problem is well known and is given by

$$W = \tfrac{1}{2}(\pi Dt)^{-\frac{1}{2}} \exp\left[-\frac{c}{2D}(x-x_o) - \frac{c^2 t}{4D}\right] \cdot \exp\left[\frac{-(x-x_o)^2}{4Dt}\right] + \exp\left[\frac{-(x-x_o)^2}{4Dt}\right] + \frac{c}{2D}\exp\left[\frac{-cx}{D}\right] \operatorname{erfc}\left[\tfrac{1}{2}(x+x_o-ct)(Dt)^{-\frac{1}{2}}\right] \qquad (1)$$

where D is the diffusion coefficient and c the drift coefficient; the barrier is sited at the origin and $x = x_o$ is the site of the introduction of unit material at $t = 0$. As $t \to \infty$, $W \to c/D \exp(-cx/D)$

and the ratio c/D is available for measurement. For large x_o and sufficiently large t, the error function term in (1) can be neglected as can the second Gaussian term to give

$$W \simeq \tfrac{1}{2}(\pi Dt)^{-\frac{1}{2}} \exp\left[\frac{-(x-x_o+ct)^2}{4Dt}\right] \quad (2)$$

from which

$$\mu = x_o - ct \quad \text{and} \quad \sigma = 2(Dt)^{\frac{1}{2}} \quad (3)$$

thus allowing for the separate estimation of the two coefficients (Chandrasekhar, 1943, p. 57).

The employment of these equations implies that a finite amount of material was introduced over a short interval at a known time in the past and at a known location. This considerably restricts possible field opportunities. Under natural conditions a continuous supply of material is more likely and if we can assume steady conditions this eliminates the need for knowledge of past conditions. However, an infinite reflective barrier as in (1) will fail to give a steady state solution. In practice barriers are not of infinite extent and if we allow for flow around a barrier then it is possible for a steady state solution to exist under conditions of continuous supply.

The problem is reduced to its simplest form if we imagine an infinite circular cylinder perpendicular to the direction of flow. This can be formalized into: two-dimensional flow subject to the diffusion equation, in an infinite region, initially at value V_o, moving with uniform velocity U in the x direction and where there is no flux across the boundary $x^2 + y^2 = a^2$. A solution is given in outline in Carslaw & Jaeger (1959) for the constant value boundary conditions, following upon earlier work by Goldstein (1932), Carslaw & Jaeger (1940) and Jaeger (1942); but the first full investigation of this type of problem, for the zero boundary condition, in a medium at rest, is that of Nicholson (1922). No published version of the precise problem stated above is known but it is most probable that it has been solved before.

The solution comprises a steady state term

$$V = V_o + V_o \frac{U}{K} \exp\left[-U_r \cos\theta\right] \times$$

$$\times \sum_0^{\infty} \varepsilon_n \frac{I_n(U_a)\, K_n(U_r) \cos n\theta}{\left[\frac{U}{4K} K_{n+1}(U_a) - h\, K_n(U_a) + \frac{U}{4K} K_{n-1}(U_a)\right]} \quad (4)$$

where $I_n(z)$ and $K_n(z)$ represent the first and second kinds of modified Bessel functions of the nth order and $U_a = Ua/2K$, $U_r = Ur/2K$, $h = U/K\ (1-\tfrac{1}{2}\cos\theta)$ and $\varepsilon_o = 1$, $\varepsilon_n = 2$, $n \geq 1$. Plus a transient term that has to remain as a complicated integral. Solutions in asymptotic series of modified Bessel functions are available for very small and very long times.

In our problem, a is the radius of the cylindrical barrier, say a telegraph pole; K the diffusion coefficient; U the drift velocity downslope and V the concentration of soil particles. In

Fig. 1 we illustrate the form of the steady state solution. Even if this were the whole of the story the disentanglement of values for U and K from field replications of Fig. 1 would be a matter of some difficulty as there is no simpler relationship than (4).

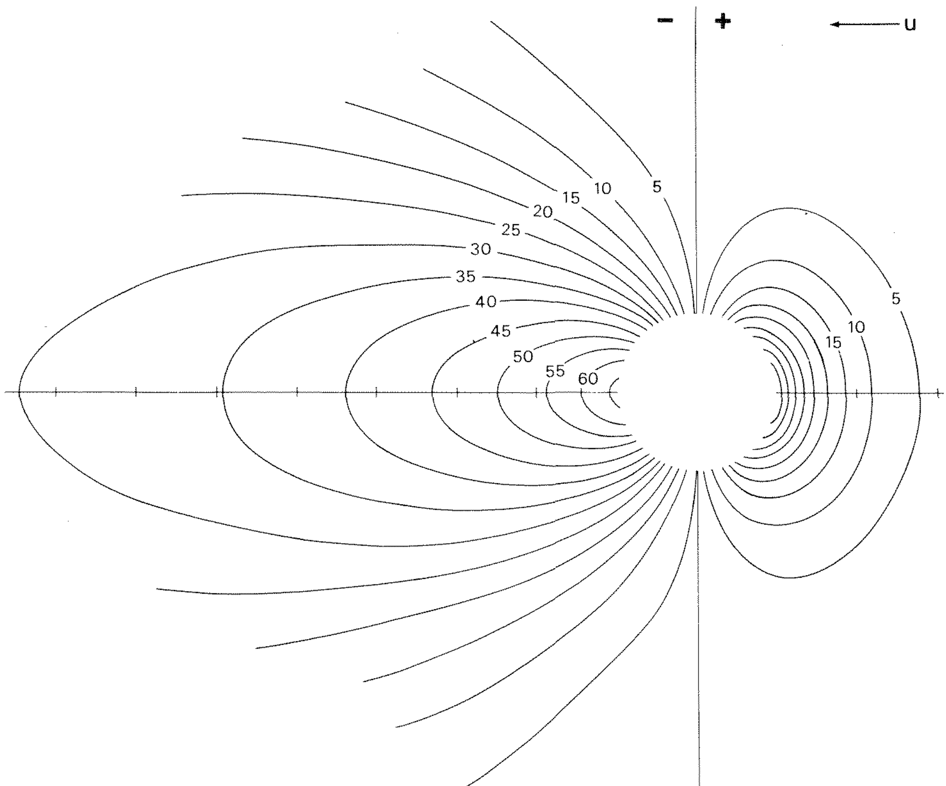

Fig. 1 Steady state solution according to equation (4).

If we were dealing with heat conduction where diffusive flow takes place without direct effect upon the structure of the medium it would be the whole, but in soil transport the transported factor itself provides the structure. The diffusion will be concentration dependent and there may arise, depending upon the kind of soil, problems of restructuring and consolidation. Concentration dependent diffusion poses two major problems: a theoretical treatment of the relationship between the diffusion coefficient and particle density or failing this a reasonable hypothetical model to assess the modification to Fig. 1 to be looked for in the field; and secondly the numerical solution of the resulting equation using possibly a Boussinesq type distribution as a starter; for an analytical treatment is out of the question.

The investigation has reached the stage when feedback from field observation is urgently required. Despite the difficulties, both theoretical and practical, the steady state solution method of measurement is worth persevering with as it offers the inestimable advantage of a single, completely destructive observation; in fact it can be carried out in the laboratory.

VELOCITY ESTIMATION BY COUNTING

In this approach we take advantage of Smoluchowski's (1916) analysis of the temporal fluctuations in the number of particles to be found within a well defined region under conditions of diffusion equilibrium. Chandrasekhar (1943), in a celebrated review, gives a detailed account of the Smoluchowski theory as well as the experimental investigations of Svedberg and Westgren. Subsequent work (partly independent) by Furth (1918, 1919) on traffic processes and much later by Rothschild (1953) on the speed of spermatozoa extended the scope of the theory to nonequilibrium conditions. Further extensions are due to Lindley (1954) and the theory has been given a rigorous formulation by Kac (1959). Recent work by Ruben (1964) has generalized the theory to cover partitioned regions which provide a more stringent method of velocity estimation. More recently McDonnough (1978, 1979) has cleared up the problem of the precise relationship between the Smoluchowski process and quite general Markovian and immigration-emigration processes. The outcome is that the Smoluchowski results can be extended to a far wider class than the Gaussian type processes, the only restrictions being independence and stationarity. The method is therefore not restricted to Brownian type motion as envisaged in the original theory of soil creep, nor indeed to soil transport processes but applies to particularate transport processes in general.

The essence of the method resides in Smoluchowski's investigation of the strictly stationary stochastic process $\{n_R(t);\ t \gtrsim 0\}$, defining the number of particles to be observed in the region R during $t \gtrsim 0$. From his theory of the probability after-effect $(1-P_R(t))$ (where $P_R(t)$ would now be termed the autocorrelation function for the process), Smoluchowski deduced that

$$E\left[(n(t+\tau) - n(t))^2\right] = 2\nu_R\ (1-P_R(\tau)) \tag{5}$$

where $n(t)$ and $n(t+\tau)$ are the numbers observed in the region R at times t and $t+\tau$ and where ν_R is the average number of particles found in R. From a series of observations we can estimate the expectation in (5) from

$$\sum_{i=1}^{m=1} \left[1/(m-1)\right]\ \left[n_{i+1}-n_i\right]^2 \tag{6}$$

We also have

$$\nu = \bar{n} = \sum_{i=i}^{m} n_i/m$$

and so we can estimate $P_R(\tau)$ which is related to the velocity of the particle both diffusive and drift.

By treating the process in an elementary manner as of Birth and Death type, Smoluchowski gives the transition probabilities as

$$W(n,n+k) = e^{-wp} \sum_{i=0}^{n} C_i^{\,n}\ p^i (1-p)^{n-i}\ \frac{(wp)^{i+k}}{(i+k)!} \tag{7}$$

$$W(n,n-k) = e^{-wp} \sum_{i=0}^{n} C_i^{\,n}\ p^i (1-p)^{n-i}\ \frac{(wp)^{i-k}}{(i-k)!} \tag{8}$$

where C_i^n is the Bernouilli factor $n!/i!(n-i)!$.

From which it can be shown (Chandrasekhar, 1943, p. 46) that

$$\begin{aligned} \langle\Delta n\rangle &= \langle m\rangle - n = (\nu-n)p \\ \langle\Delta n^2\rangle &= p^2\left[(\nu-n)^2-n\right] + (n+\nu)p \end{aligned} \tag{9}$$

where $m = n \pm k$.

Averaging over all values of n,

$$\begin{aligned} \langle\Delta\rangle &= \langle\langle\Delta_n\rangle\rangle = \langle\nu-n\rangle p = 0 \\ \langle\Delta^2\rangle &= \langle\langle\Delta_n{}^2\rangle\rangle = 2\nu p \end{aligned} \tag{10}$$

These latter relations provide a connection with the theory of the diffusion equation and also a means of experimental determination of the probability after-effect factor P from the evaluation of the mean square differences from a long series of observations which involve no more than the counting of the number of particles present within the region R at successive instants separated by an appropriate time interval.

Ruben (1964) has generalized the procedure by subdividing the region, $R = \Sigma R_i$. The corresponding notion to Smoluchowski's probability after-effect is the covariance matrix $\underline{P}(\tau)$ of the vector stochastic process $\underline{n}(t)$. For the simplest case where R is subdivided into halves in place of (5) we have

$$E\begin{bmatrix} \left[n_1(t+\tau)-n_1(t)\right]^2 & \left[n_1(t+\tau)-n_1(t)\right]\left[n_2(t+\tau)-n_2(t)\right] \\ \left[n_2(t+\tau)-n_2(t)\right]\left[n_1(t+\tau)-n_1(t)\right] & \left[n_2(t+\tau)-n_2(t)\right]^2 \end{bmatrix}$$

$$= 2\,\hat{\underline{N}}\left[I-\hat{\underline{P}}(\tau)\right] = 2\begin{bmatrix} \hat{\nu}_1 & \\ & \hat{\nu}_2 \end{bmatrix}\begin{bmatrix} 1-\hat{p}_{11} & 1-\hat{p}_{12} \\ 1-\hat{p}_{21} & 1-\hat{p}_{22} \end{bmatrix} \tag{11}$$

Furth (1918) adapted Smoluchowski's theory to estimate drift velocities. In a famous experiment he estimated the average speed of pedestrians by counting the number present within a given stretch of pavement at intervals of 5 s.

If v is the mean speed of a particle and h is the length in the downslope direction, then after a long series of observations the probability after-effect tends to the values

$$1-P_r(\tau) = \begin{cases} v\tau/n & \tau \leqslant h/v \\ 1 & \tau \geqslant h/v \end{cases} \tag{12}$$

$P_R(\tau)$ is then the autocorrelation function.

The background drift velocity is assumed stationary so that $P_R(\tau)$ can be estimated from (5) as before,

$$\frac{1}{m-1}\sum_{\ell=1}^{m-1}\left[n_R(\ell\tau)-n_R((\ell-1)\tau)\right]^2 = 2\bar{n}(1-\hat{P}_R(\tau)) \tag{13}$$

where

$$\bar{n} = \frac{1}{m}\sum_{s=0}^{m-1} n_R(s\tau) \tag{14}$$

Thus the mean velocity can be calculated from a sequence of counts of particles within the region R at instants separated by a selected interval τ. The generalization of Ruben can be applied to the Furth process to give a more precise experimental procedure but at the cost of a more complicated formulation.

For other than rapid transport processes the disadvantage of the requirement of a long series of observations (505 for the Furth experiments) can be partially circumvented by the use of the Ruben procedure but more importantly by the employment of an ergodic argument that allows for the replacement of the time observations by spatial observations. We deal essentially with stationary processes and the temporal fluctuation of number values within a region will tend to a Gaussian form. This can be established fairly rigorously using classical shot-effect arguments (Rice, 1944). Provided we deal with, at least, a weakly stationary process and over a selected (plane) slope we can be reasonably sure that we deal with one ensemble then certain functions associated with the process, such as the mean, mean square deviation and for higher order stationarity the autocorrelation function will possess identical values whether we take a series of observations on a selected region at set intervals, or on a set of equivalent regions spaced either along a line downslope or along one perpendicular to this across the slope.

The Smoluchowski method requires that we have a sufficient set of paired observations separated by a given interval. It is only economy of effort that dictates a consecutive series. Instead of arranging the pairs of observations on a continuous time series we can arrange them as pairs either across or down the slope. We cannot avoid making a pair of observations at any one site. This is because we have no knowledge of the scaling factor between the time interval and the equivalent spatial interval. Otherwise we could take advantage of conventional ergodic arguments and substitute a measure of the autocorrelation function from a spatial set of observations. Thus we cannot reduce the exercise to a once only observation and therefore the first observation has to be carried out without interference. This is no restriction, of course, if we are measuring a purely surface process or one which can be reduced to transparency by the use of suitable tracers.

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