

Mathematical model for reservoir silting

P. R. J. COGOLLO & S. M. VILLELA

*Escola de Engenharia de Sao Carlos, USP, Campos de Sao Carlos,
Av. Dr Carlos Botelho 1465, Cx P 359, 13560 Sao Carlos, SP, Brazil*

Abstract The construction of a dam changes the natural equilibrium of a river: the flow velocity will be reduced and the sediment transported will be deposited in the reservoir. Estimation of the reservoir sedimentation rates and location of sediment is a complex task due to many interrelated factors, such as size and texture of the sediment particles, seasonal variations in river flow and sediment load, size and shape of the reservoir and reservoir operation scheduling. This work provides a means of estimating the sediment distribution in time and space into the lake. This is the first step to compute the useful life of the reservoir.

Modèle mathématique de décantation dans une retenue

Résumé La construction d'un barrage change l'équilibre naturel d'un fleuve: la vitesse diminuera et le sédiment transporté sera déposé dans le réservoir. L'estimation du volume des dépôts et leur localisation est une tâche complexe, vu le nombre de facteurs en jeu, tels que la taille et la texture des particules, la variation saisonnière des débits liquides et solides, la taille, la forme et le mode d'opération du réservoir. Le travail ici présenté permet d'estimer la distribution des dépôts dans une retenue (tant dans le temps que dans l'espace), et constitue un premier pas pour le calcul de la vie utile d'un réservoir.

RESERVOIR SEDIMENTATION

A delta is formed in the entrance of the reservoir by coarser sediment from the bed and from suspended load. When velocity decreases, the deposit extends ever farther into the reservoir. Fig. 1 shows a scheme of the sediment routing in the reservoir.

THE MATHEMATICAL MODEL

The mathematical model used in this work was proposed by Lopez (1978). It gives the rate and pattern of the sediment deposition in a river-reservoir system. Figure 2 shows a reservoir formed by a dam. The river is modelled by a single channel assuming one-dimensional flow as predominant phenomenon whereas a set of multiple channels are used to simulate the river

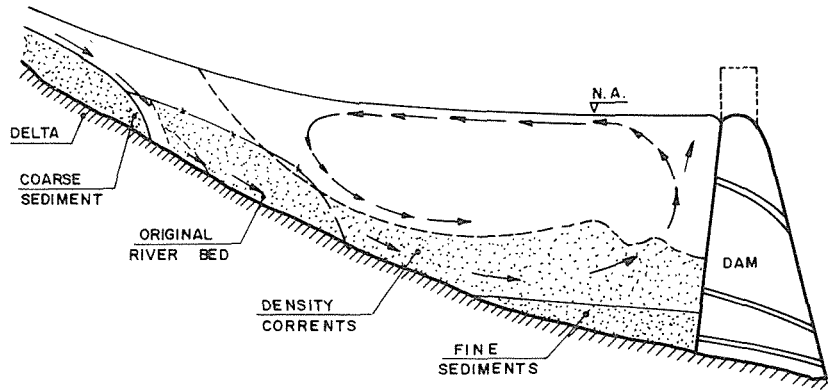


Fig 1 Typical reservoir-delta profile.

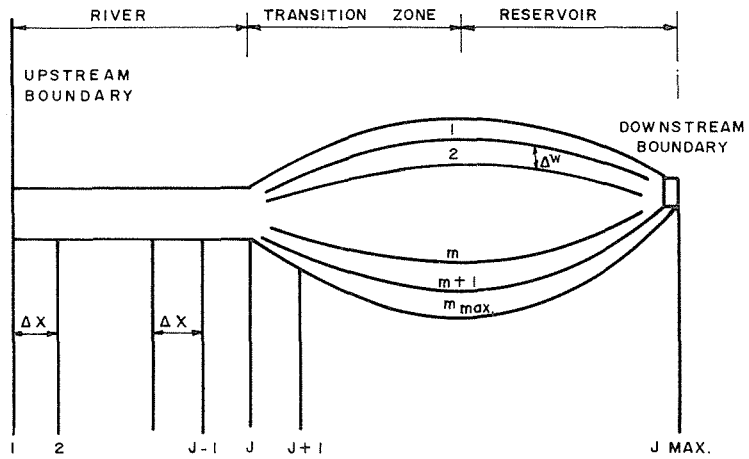


Fig. 2 The river-reservoir system.

and the flood plains in the reservoir.

The physical boundaries for the model are: the flow and sediment rates at the upstream section of the river, and stages at downstream section of the reservoir.

BASIC EQUATIONS

The basic equations used in this study are the differential equations for simulating gradually varied unsteady flow in natural alluvial channels. These equations are: the flow continuity and the flow momentum equation, and the sediment continuity equation. (Symbols are defined in the Appendix to the paper.)

The continuity equation for sediment-laden water is:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} + \frac{\partial A_d}{\partial t} - q_l = 0 \quad (1)$$

The momentum equation for sediment-laden water is:

$$\frac{\partial(\beta QV)}{\partial x} + \frac{\partial Q}{\partial t} + gA \frac{\partial Y}{\partial x} = gA(S_o - S_f) + q_l V_l \quad (2)$$

The sediment continuity equation is given by:

$$\frac{\partial Q_s}{\partial x} + \frac{\partial A_d}{\partial t} + \frac{\partial(AC_s)}{\partial t} - q_s = 0 \quad (3)$$

A definition sketch for one-dimensional flow movement in natural channels is given in Fig. 3.

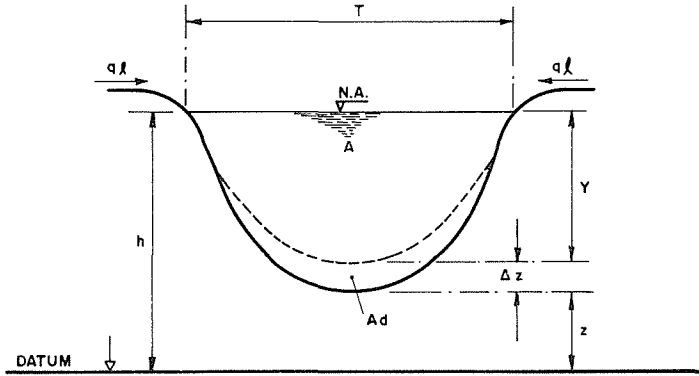


Fig. 3 Sectional view.

The following assumptions are made in deriving those equations:

- The channel is sufficiently straight and uniform in the river reach for the flow characteristics to be physically represented by a one-dimensional model.
- The flow velocity is uniformly distributed over the cross section.
- Hydrostatic pressure prevails at every point in the channel flow.
- The water surface slope is small.
- The density of the sediment-laden water is constant over the cross section.
- The unsteady flow resistance coefficient is assumed to be the same as for steady flow in alluvial channels.

The basic equations (1), (2) and (3), form a set of nonlinear first-order partial differential equations. They have two independent variables, x and t , and three dependent ones Q , Y and A_d . The remaining variables must be expressed as functions of those three basic unknowns in order to obtain the solution.

NUMERICAL ANALYSIS

Herein, the partial derivatives of the flow continuity and flow momentum equations are approximated by a linear-implicit scheme of finite differences whereas an explicit scheme is used for solving the sediment continuity equation.

SOLUTION ALGORITHM

According to Lopez (1978), equations (1) and (2), after discretization, can be simplified, to:

$$a Q_{j+1} + b Q_j + c Y_{j+1} + d Y_j + e = 0$$

$$a' Q_{j+1} + b' Q_j + c' Y_{j+1} + d' Y_j + e' = 0$$

where Q_{j+1} , Q_j , Y_{j+1} and Y_j are unknowns and all the coefficients are functions of known quantities. Then, the double-sweep method of solution, as applied by Liggett & Cunge (1974) is used to solve the system of linear algebraic equations. After solving for Q_j and Y_j for $j = 1, \dots, j_{\max}$, the next step is to determine the value of these variables at each individual subsection ($Q(m,j)$ and $Y(m,j)$), as follows:

- A parabolic jet is assumed, having width given by:

$$w = a \sqrt{x} + B_o$$

- In the reservoir there are two zones, one of transition and the other of flow establishment. The limit of the zone of flow establishment can be obtained by:

$$x_o = (B_o/a)^2$$

In the transition zone the flow discharge is given by:

$$Q(m,j) = \int_0^{w_1} V_0 dA + \int_0^{w_2} V_x dA$$

where:

$$V_x = V_0(3\sqrt{x} - B_o - 3w)/(2a\sqrt{x})$$

and in the flow establishment zone the flow discharge is:

$$Q(m,j) = \int_{w_3}^{w_4} V_x dA$$

where:

$$V_x = V_0 2B_0 (a \sqrt{x} - B_0 - 2w) / (a \sqrt{x} + B_0)$$

After having the $Q(m,j)$ by the equations above, the process of sedimentation is given by the sediment continuity equation:

$$\begin{aligned} \Delta A_d = & \left[\frac{4\Delta t}{\gamma_{d_{m_{j-1}}}^{n+1} + \gamma_{d_{m_{j-1}}}^n + \gamma_{d_{m_j}}^{n+1} + \gamma_{d_{m_j}}^n} \right] \cdot \left[-\frac{1}{2\Delta x} \left[Q_{s_{m_j}}^{n+1} - Q_{s_{m_{j-1}}}^{n+1} + \right. \right. \\ & \left. \left. + Q_{s_{m_j}}^n - Q_{s_{m_{j-1}}}^n \right] - \frac{1}{2\Delta x} \left[A_m \cdot C_{s_m} \right]_j^{n+1} - \left[A_m \cdot C_{s_m} \right]_j^n - \right. \\ & - \frac{1}{4} \left[q_{s_{(n-1)j-1}}^{n+1} + q_{s_{(n-1)j-1}}^n + q_{s_{(n-1)j}}^{n+1} + q_{s_{(n-1)j}}^n \right] + \frac{1}{4} \left[q_{s_{m_{j-1}}}^{n+1} + \right. \\ & \left. \left. + q_{s_{m_{j-1}}}^n + q_{s_{m_j}}^{n+1} + q_{s_{m_j}}^n \right] \right] \end{aligned}$$

The change in the bed elevation at any cross section can be approximated by:

$$Dz(m,j) = \Delta A_d(m,j) / (T(m,j))$$

where

$$\Delta A_d = Ad_{m_j}^{n+1} - Ad_{m_j}^n$$

The original bed material granulometry is obtained from direct measurements. The granulometry subsequent intervals of time and for all the sections can be obtained from the fraction of sediment deposited values, as follows:

$$\begin{aligned} Fb_i &= (Z_i^{n-1} + Dz_i^n) / (d^{n-1} + \sum_{i=1}^N Dz_i^n) & \text{for } \begin{cases} i = 1, \dots, NG \\ i = 1, \dots, N_{\max} \end{cases} \\ ds^{n-1} &= ds^{n-2} + \sum_{i=1}^N Dz_i^{n-1} & \text{for } n = 2, \dots, N_{\max} \\ Z_i^{n-1} &= Fb_i^{n-1} \cdot ds^{n-1} & \text{for } n = 2, \dots, N_{\max} \end{aligned}$$

with the initial condition:

$$Z_i^0 = Fb_i^0 + ds^0$$

The lateral inflow of sediment has to be determined from erosion equations outside this model. In the example, a constant rate of 12.5 t ha⁻¹ year⁻¹ was used.

The specific weight of sediment deposit after t years can be computed, according to Miller (1953) by:

$$\gamma_{d_m} = \gamma_{d_i} + 0.434K \left[\frac{t}{t-1} \log_e t - 1 \right]$$

where γ_{d_i} is given by the procedure of Lara & Pemberton (1963). A simplified scheme of the mathematical model is shown in Fig. 4.

RESULTS

The model was applied to the reservoir of Urra II in the Sinu River, in Colombia. The results are given in Table 1 and Fig. 5.

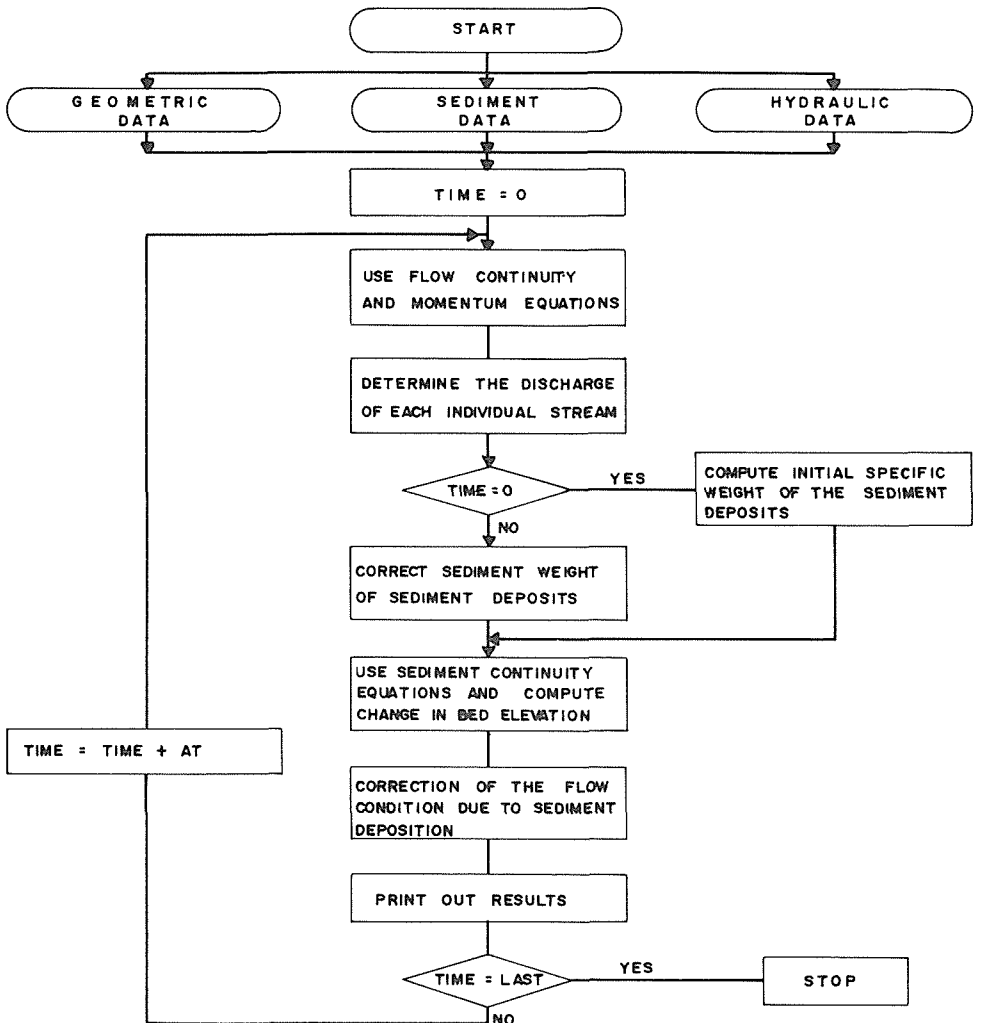


Fig. 4 Flow chart of the mathematical model.

Table 1 Variation of bed elevation with time — Urra II

Section	Channel	Initial	Bed elevation (m):			
			50 years	100 years	150 years	200 years
1	1	1.0	1.0	1.0	1.0	1.0
1	2	1.0	1.0	1.0	1.0	1.0
1	3	1.0	1.0	1.0	1.0	1.0
2	1	1.0	1.0	1.0	1.0	1.0
2	2	1.0	1.0	1.0	1.0	1.0
2	3	1.0	1.0	1.0	1.0	1.0
3	1	1.0	3.9	6.5	6.7	7.0
3	2	1.0	5.8	10.0	10.2	10.5
3	3	1.0	4.1	6.7	7.0	7.4
4	1	1.0	1.6	1.9	4.1	5.1
4	2	1.0	2.1	2.5	5.2	6.4
4	3	1.0	1.7	1.9	4.2	5.2
5	1	1.0	1.2	1.4	1.6	2.1
5	2	1.0	1.3	1.5	1.7	2.1
5	3	1.0	1.0	1.4	1.7	2.1
6	1	1.0	1.0	1.5	1.5	1.9
6	2	1.0	1.0	1.5	1.6	1.8
6	3	1.0	1.0	1.4	1.6	1.9
7	1	1.0	1.0	1.0	1.1	1.2
7	2	1.0	1.0	1.0	1.0	1.2
7	3	1.0	1.0	1.0	1.1	1.3
8	1	1.0	1.0	1.0	1.0	1.0
8	2	1.0	1.0	1.0	1.0	1.0
8	3	1.0	1.0	1.0	1.0	1.0

CONCLUSIONS

The results were considered satisfactory at least from a qualitative point of view. A total calibration of the model could be made with bed elevations surveyed on at least two occasions. Figure 5 shows the results of the model runs for the reservoir of Urra II. It can be seen that almost all the sediment is deposited in the entrance of the reservoir. This result is in agreement with the delta pattern described by Graf (1971).

The jet theory in this study assumes that the boundary of the jet spreads parabolically from the inlet section of the reservoir. For the case of a reservoir with multiple inlets, the interaction between jets must also be studied in the future.

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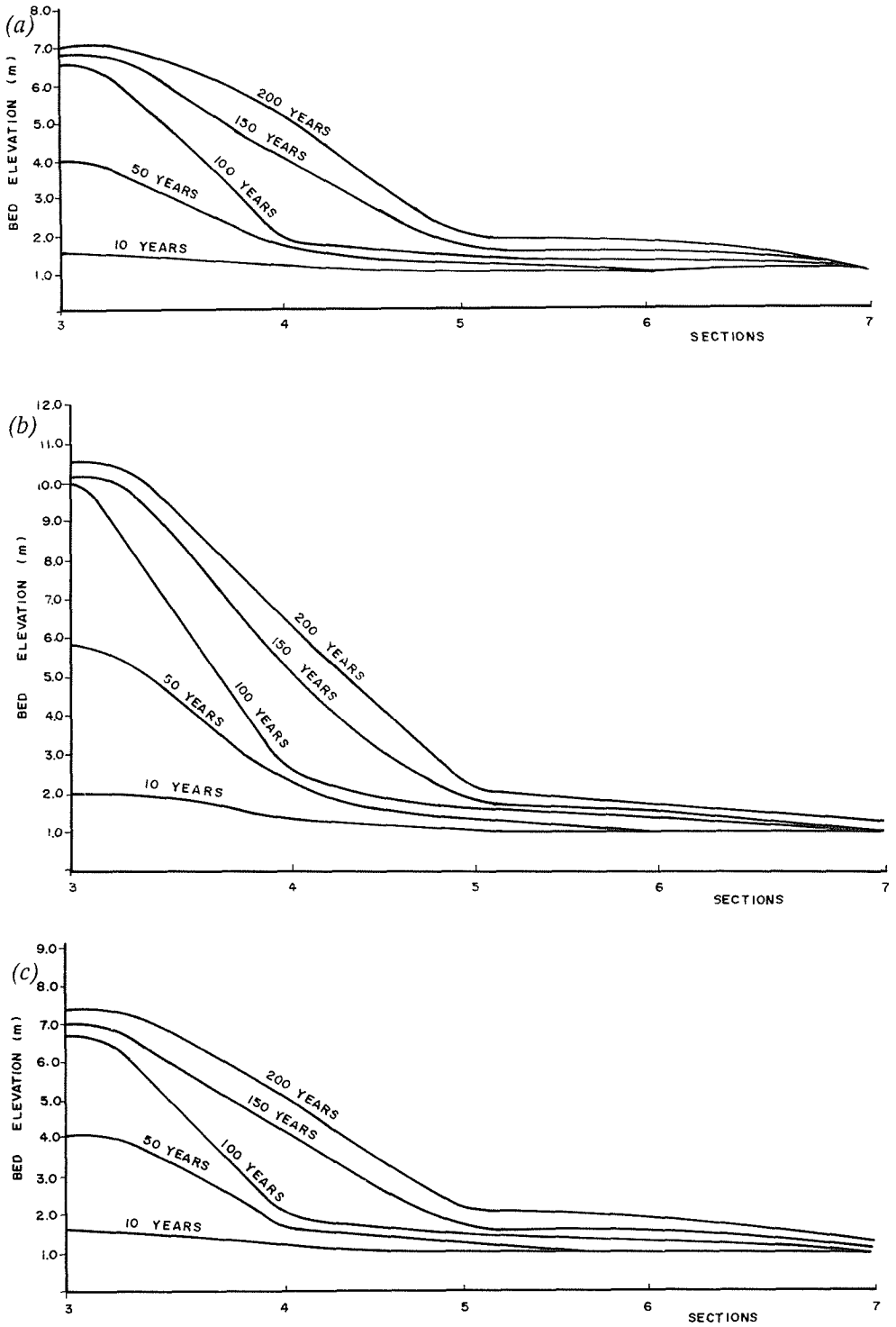


Fig. 5 Sedimentation in (a) subchannel no. 1, (b) subchannel no. 2, and (c) subchannel no. 3 — Urra II reservoir.

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APPENDIX: NOTATION

- A cross-sectional area of the channel (m^2)
 a coefficient, depends on the river-reservoir system
 A_d volume of deposition or erosion of sediment on unit length of channel bed (m^2)
 A_m cross-sectional area of the subchannel (m^2)
 B_o top width of the river (m)
 C_s average sediment concentration in the cross section on a volume basis
 C_{s_m} average sediment concentration ($t\ m^{-3}$)
 Dz_i bed elevation change of size fraction i at time step n (m)
 ds total sediment depth (m)
 Fb_i fraction of the i th size in the bed material
 g gravitational acceleration ($m\ s^{-2}$)
 NG number of grain sizes
 N_{max} number of time steps
 P volume of sediment in a unit volume of bed layer $P = \gamma_d / \gamma_s$
 Q discharge of sediment-laden water ($m^3\ s^{-1}$)
 $Q(m,j)$ flow discharge in subchannel m and section j ($m^3\ s^{-1}$)
 Q_s total sediment load ($m^3\ s^{-1}$)
 Q_{s_m} total sediment load in the subchannel ($t\ s^{-1}$)
 q_l lateral inflow of sediment-laden water ($m^2\ s^{-1}$)
 q_s lateral sediment inflow into the stream ($m^2\ s^{-1}$)
 q_{s_m} lateral transfer of sediment ($t\ m^{-1}\ s^{-1}$)
 S_f friction slope
 S_o bed slope
 t time
 $T(m,j)$ top width for the stream m and section j (m)
 V mean flow velocity ($m\ s^{-1}$)
 V_l velocity component of the lateral flow in the x -direction ($m\ s^{-1}$)
 V_o velocity at the inlet of the reservoir ($m\ s^{-1}$)
 w variable of integration (m)
 x distance from the inlet of the reservoir (m)
 x_o distance from the inlet section to the end of the zone of flow establishment (m)
 Y flow depth (m)
 Z_i available sediment depth of i th fraction of the bed material (m)
 β momentum correction factor for velocity distribution ($t\ m^{-3}$)
 γ_d specific weight of the sediment deposits ($t\ m^{-3}$)
 γ_{s_m} specific weight of the sediment particles ($t\ m^{-3}$)

